COMPUTATION OF AFFINE TIME-FREQUENCY DISTRIBUTIONS USING THE FAST MELLIN TRANSFORM

J.P.Ovarlez (*), *J.Bertrand* (**) *and P.Bertrand* (*)

(*) ONERA, BP 72, 92322 Chatillon, France
(**) CNRS and University Paris VII, 75251 Paris, France

ABSTRACT

Theoretical approach of the broad-band time-frequency problems has led to consider new time-frequency distributions affiliated with the affine group of clock changes. Due to their origin these distributions involve stretched forms of the signals which are not easy to compute by standard techniques. The object of the paper is to solve this difficulty by giving efficient algorithms founded on the use of the Fast Mellin Transform.

1. INTRODUCTION.

The class of affine time-frequency distributions is the outcome of a theoretical investigation concerning the interpretation of non-stationary broad-band signals ([1] for a review). This class consists of bilinear functionals of the signal and is connected with the affine group in the same way as Cohen's class [2] is with Heisenberg's group. Specific forms have been selected which generalize in a certain sense the Wigner-Ville distribution and it is the purpose of this paper to give a practical computational procedure for these forms.

Recall that the affine group of time dilations $a > 0$ and translations $b \in R$ acts upon the physical signal $s(t)$ as:

$$s(t) \longrightarrow a^r \, s(a^{-1}(t-b)) \qquad (1)$$

If the Fourier transform $S(f)$ of the signal $s(t)$ is defined by:

$$S(f) = \int_{-\infty}^{\infty} e^{-2i\pi f t} s(t) \, dt \qquad (2)$$

the action reads:

$$S(f) \rightarrow S_{a,b}(f) = a^{r+1} e^{-2i\pi b f} S(af) \qquad (3)$$

The real number r is related to the physical dimension of the signal. A standard choice however is $r = -1/2$.

Transformation (3) requires to introduce the invariant scalar product:

$$(S, S') = \int_{-\infty}^{\infty} S(f) S'^{*}(f) \mid f \mid^{2r+1} \, df \qquad (4)$$

The affine family of greatest potential interest for signal theory is the so-called diagonal subclass given by [3]:

$$P(t,f) = \mid f \mid^{2r-q+2} \int_{-\infty}^{\infty} e^{2i\pi t f(\lambda(u)-\lambda(-u))} S(f\lambda(u)) S^{*}(f\lambda(-u)) \mu(u) \, du \qquad (5)$$

In this expression, function μ is fairly arbitrary while function λ, beside being positive, satisfies a few technical conditions. Specific forms of λ which lead to localization properties are [1]:

$$\lambda_k(u) = \left(k \frac{e^{-u} - 1}{e^{-ku} - 1} \right)^{\frac{1}{k-1}}, \quad k \leq 0 \qquad (6)$$

The parameter q occurring in (5) is a real number which depends on the meaning of P. The classical choice $q = 0$ is consistent with the interpretation of P as a sort of probability law. The greatest number of properties, including Moyal's relation and localization, is obtained for the limiting case $k = 0$ in (6) and a particular function μ. The corresponding distribution which will be noted P_0 is given by:

$$P_0(t,f) = \mid f \mid^{2r-q+2} \int_{-\infty}^{\infty} e^{2i\pi t f u} \times \qquad (7)$$
$$S\left(f \frac{u e^{u/2}}{2 \sinh u/2}\right) S^{*}\left(f \frac{u e^{-u/2}}{2 \sinh u/2}\right) \left(\frac{u}{2 \sinh u/2}\right)^{2r+2} du$$

From a computational point of view this formula may look impressive. In fact the use of an adequate Mellin transform reduces the task of its implementation to just a few FFT. The method can also be used for computing any affine distribution in the diagonal subclass (5).

In Section 2, we recall the form of the discrete Mellin transform connecting geometric samples of the signal $S(f)$ to arithmetic samples of its Mellin transform [4]. The application of sampling conditions is discussed and a practical procedure is proposed. An algorithm for the computation of the diagonal affine distributions is given in Section 3. Finally , in Section 4 some examples of application to mathematical signals are presented and discussed.

2. THE MELLIN TRANSFORM AND ITS DISCRETIZATION.

0-7803-0532-9/92 $3.00 © 1992 IEEE

Expression (5), when written with a positive $\lambda(u)$ function, does not involve any cross terms between positive and negative frequency parts of the signal. This property, combined with the fact that the distribution associated with a real signals $s(t)$ is an even function of the f variable, permits to consider the implementation of (5) for analytic signals only.

The Mellin transform we use operates on the analytic signal $Z(f)$ by the formula:

$$M^{\xi}[Z](\beta) = \int_0^{\infty} Z(f)\, e^{2i\pi\xi f}\, f^{2i\pi\beta + r}\, df \qquad (8)$$

and can be inverted by:

$$Z(f) = \int_{-\infty}^{\infty} M^{\xi}[Z](\beta)\, e^{-2i\pi\xi f}\, f^{-2i\pi\beta - r - 1}\, d\beta \qquad (9)$$

In these expressions ξ represents a real parameter which will be interpreted below. The basic feature of transformation (8) appears when considering the action of a dilation of factor a ($a > 0$) around a chosen time ξ. In fact, this operation which is defined by:

$$Z(f) \longrightarrow Z'(f) = a^{r+1} e^{-2i\pi\xi(1-a)f}\, Z(af) \qquad (10)$$

is simply represented in Mellin's space by the multiplication:

$$M^{\xi}[Z](\beta) \longrightarrow M^{\xi}[Z'](\beta) = a^{-2i\pi\beta}\, M^{\xi}[Z](\beta) \qquad (11)$$

Other properties that are essential in the following are Parseval's formula and the relations between multiplication and convolution in the two spaces [4]. In particular, we have:

$$M^{\xi}[f^{r+1} e^{2i\pi\xi f}\, Z_1\, Z_2](\beta) = (M^{\xi}[Z_1] * M^{\xi}[Z_2])(\beta) \qquad (12)$$

where the $*$-operation is a convolution in β. The Discrete Mellin transform (DMT) concerns signals with limited extension both in the f and β variables. In a general manner it relates N geometrically spaced samples of the signal in frequency to N arithmetically spaced samples of its Mellin transform. If the signal Z is limited to the band (f_1, f_2) and if the support of its Mellin transform is (β_1, β_2), then the DMT is given by [4]:

$$M^{\xi}\left(\frac{p}{N \ln \nu}\right) = \sum_{k=0}^{N-1} \left[\nu^{k(r+1)}\, e^{2i\pi\xi f_1 \nu^k}\, Z(f_1 \nu^k) \ln \nu\right] e^{2i\pi k p/N} \qquad (13)$$

This is a linear system characterized by a ratio ν and a number N which must verify the two conditions:

$$\nu^N > f_2/f_1 \quad , \quad (\ln \nu)^{-1} > | \beta_2 - \beta_1 | \qquad (14)$$

in order to avoid aliasing. As a consequence, the number N of complex samples to deal with must be no less than:

$$N = | \beta_2 - \beta_1 | \ln \frac{f_2}{f_1} \qquad (15)$$

As seen on formula (13), the efficient computation of a DMT can be performed by recycling any FFT algorithm.

There is in general no a priori knowledge of the support of the Mellin transform of a signal. However, as noted in [4], it can be asserted that the Mellin transform of any band limited signal of finite duration has a bounded support which can be determined directly. This operation is founded on the time-frequency interpretation of the Mellin variable given by (7). For example, if the signal has supports $(\xi - T/2, \xi + T/2)$ in time and (f_1, f_2) in frequency, it will be located in the time-frequency half-plane between the two hyperbolas:

$$t = \pm\frac{\beta_0}{f} + \xi \ , \ \beta_0 = \frac{T}{2} f_2 \qquad (16)$$

Such a simple geometrical analysis is sufficient to assert that, in this case,the support of the ξ-Mellin transform of the signal will be the interval $(-\beta_0, \beta_0)$.

In fact the above analysis is not only useful for estimating the support of the ξ-Mellin transform of a given signal but also for choosing a value of the parameter ξ which minimizes the size of this support. Such an operation is important in order to limit the number (15) of samples to consider in the applications.

3. COMPUTATION OF AFFINE TIME-FREQUENCY DISTRIBUTIONS

The Mellin transform (8)-(9) is of great help in the task of implementing expressions of the form (5). This fact will be illustrated first with the computation of the particular distribution (7). Hints will then be given for the extension to the general form (5).

For economical reasons we systematically use the Mellin transform (8) corresponding to $\xi = 0$ and denote it by M. The procedure will thus operate efficiently when applied to signals located about time $t = 0$. In other cases the algorithm will have to be inserted between two opposite time translations acting respectively on the signal and on its time-frequency representation.

3.1 Principle of the computation of P_0.

First the half-plane ($f > 0$) is reparametrized by setting $\gamma = ft$ and the notation $\tilde{P}_0$ is introduced by:

$$\tilde{P}_0(\gamma, f) = P_0(t, f) \qquad (17)$$

Then expression (7) is rewritten as:

$$\tilde{P}_0(\gamma, f) = f^{r+1-q} \int_{-\infty}^{\infty} [\lambda_0(u)\lambda_0(-u)]^{r+1}\, f^{r+1}\, Z(f\lambda_0(u)) Z^*(f\lambda_0(-u))\, e^{2i\pi\gamma u}\, du \qquad (18)$$

with

$$\lambda_0(u) = \frac{u e^{u/2}}{2 \sinh u/2}$$

After multiplication of this equation by f^{-r-1+q}, a Mellin transform with respect to f is performed. Thanks to relations (11)-(12) the result becomes:

$$M[f^{-r-1+q}\,\tilde{P}_0(\gamma,f)](\beta) = \int_{-\infty}^{\infty} e^{2i\pi\gamma u} \times \left[\int_{-\infty}^{\infty} X(\beta',u)X^*(\beta'-\beta,-u)\,d\beta'\right]\,du \quad (19)$$

where

$$X(\beta,u) = \lambda_0(u)^{-2i\pi\beta}\,M[Z](\beta) \quad (20)$$

The expression inside brackets in (19) is a cross-correlation which can be computed in terms of the Fourier transform $F_{\pm}(\theta,u)$ of $X(\beta,\pm u)$. The result leads to a new form of (19) which is written:

$$M[f^{-r-1+q}\,\tilde{P}_0(\gamma,f)](\beta)] = \int_{-\infty}^{\infty} e^{2i\pi\gamma u} \times \left[\int_{-\infty}^{\infty} F_+(\theta,u)F_-^*(\theta,u)e^{2i\pi\beta\theta}\,d\theta\right]\,du \quad (21)$$

Finally, inverting the Mellin transform by (9) and switching back to the time variable yields the formula to be discretized:

$$P_0(t,f) = 2\Re\left[f^{-q}\int_0^{\infty} F_+(\ln f,u)\,F_-^*(\ln f,u)e^{2i\pi tfu}\,du\right] \quad (22)$$

where $\Re$ denotes the real part operation.

3.2. Algorithm

There are two main steps in the discretization of (22), namely the computation of $F_{\pm}$ and the Fourier transform with respect to u. The domain $(0,u_0)$ of u is such that both $\lambda(u)$ and $\lambda(-u)$ stay within the interval $(f_1/f_2, f_2/f_1)$. Once u_0 is determined, the number M of samples in u is chosen such that:

$$M \geq BT\frac{a}{a-1}u_0\ ,\ a \equiv f_2/f_1 \quad (23)$$

Step 1: Suppose we start with a signal Z geometrically sampled on (f_1,f_2) with the ratio $\nu = (f_2/f_1)^{1/N}$, where N is the number of points. For any $u = nu_0/M$, $0 \leq n \leq M-1$, compute the Fourier transform of X given by (20) with respect to β. Since the result is a convolution, the number of samples in β-space must be doubled to avoid aliasing. This is achieved by padding the signal with N zeros in f-space. Functions $F_{\pm}$ are then obtained by discrete FFT on X. Here we notice that only N samples will be significant. The constraint on N comes from the Mellin transformation and is identical with (15) which reads here:

$$N \geq BT\frac{a\ln a}{a-1} \quad (24)$$

Step 2: Perform the u-Fourier transform in (22) which is expressed by the discrete formula:

$$\tilde{P}[k,p] = f_1^{-q}\nu^{-qp}\sum_{n=0}^{M-1} F_+[p,n]F_-^*[p,n]e^{2i\pi kn/M} \quad (25)$$

where $0 \leq k \leq M-1, 0 \leq p \leq N-1$.

The approximate complexity of this algorithm can be expressed in terms of the number of FFT performed. If the time-frequency representation $P_0(t,f)$ is characterized by (M,N) points respectively in time and frequency, we have to deal with $(2M+1)$ FFT of $2N$ points and (N) FFT of M points.

In the limit of narrow-band signals, relations (23) and (24) are reduced to $M \geq 2BT$ and $N \geq BT$ respectively. This result can be compared with what is usually obtained when computing Wigner-Ville's function [5] [6].

3.3. Computation of distributions of the form (5)

For these distributions the u-integral in the definition formula is no longer a Fourier transform. This property can however be recovered by a change of variable whenever function $\Phi(u) = \lambda(u) - \lambda(-u)$ is monotonous. This is so, in particular, for functions in family (6). In such a case, performing explicitly the change of variable $v = \Phi(u)$ in (5) (written with $Z(f)$) allows the rewriting of this equation under the form:

$$\tilde{P}(\gamma,f) = P(t,f) = f^{r+1-q}\int_{-\infty}^{\infty}\left[\tilde{\lambda}(v)\tilde{\lambda}(-v)\right]^{r+1} f^{r+1}Z(f\tilde{\lambda}(v))Z^*(f\tilde{\lambda}(-v))e^{2i\pi\gamma v}\tilde{\mu}(v)dv \quad (26)$$

where the notations are: $\gamma = tf$, $\tilde{\lambda}(v) = \lambda[\Phi^{-1}(v)]$ and where the odd-parity property of Φ^{-1} has been used explicitly. The new function $\tilde{\mu}$ is obtained from μ by taking into account the change of integration measure and the factor $(\tilde{\lambda}(v)\tilde{\lambda}(-v))^{-r-1}$.

The computation algorithm for (26) is quite similar to the algorithm for P_0. The main change comes from the existence of the extra factor $\tilde{\mu}(v)$ in the final Fourier transform. Thus the numerical computation of any distribution (5) involves basically the same kind of work. The only additional problem in the general case is the inversion of function Φ. Although such an operation is not always easy analytically it presents no difficulty numerically and does not actually add to the numerical work.

4. APPLICATION TO PARTICULAR SIGNALS

The above procedure has been applied to signals described by mathematical functions of the frequency. In this case, the geometric sampling of the signal is easy to obtain and the algorithm can be applied directly. Figures 1 to 3 are P_0-representations involving hyperbolic signals of the form:

$$Z(f) = f^{-r-1}e^{-2i\pi\alpha\ln f} \quad (27)$$

The P_0 representation of this signal is mathematically localized on a hyperbola in the time-frequency half-plane. Figure 1 gives the numerical form of the result. If the analyzed signal is composed with two hyperbolic signals, cross-terms occur which can be observed on Fig. 2 and 3.

The P_0 representation of a narrow-band signal is very close to its Wigner-Ville representation. This property can be verified on Fig.4 which gives the P_0-representation of a chirp (linear group delay modulation).

5. CONCLUSION

Computation of the affine time-frequency distributions can be efficiently performed with the help of the Discrete Mellin Transform. The algorithm involves only FFT routines and runs very fast. This implementation allows to consider the time-frequency representations as practical tools for the study of broad-band signals.

References

[1] J.Bertrand and P.Bertrand, "Affine time-frequency distributions" in " Time-Frequency Signal Analysis - Methods and Applications", ed. B.Boashash, Longman-Cheshire,1991.

[2] L.Cohen, " Time-frequency distributions - A review" , Proc. IEEE vol.77, pp. 941-981, 1989.

[3] J.Bertrand and P.Bertrand "Time-frequency representations of broad-band signals", Proc.IEEE ICASSP-88, pp.2196-2199, 1988.

[4] J.Bertrand, P.Bertrand and J.P.Ovarlez, " Discrete Mellin transform for signal analysis", Proc. IEEE ICASSP-90, pp.1603-1606, 1990.

[5] F.Peyrin and R.Prost, IEEE trans. ASSP, **34**, pp.858-867, 1986.

[6] B.Boashash and P.Black, IEEE Trans.ASSP, **35**, pp.1611-1618, 1987.

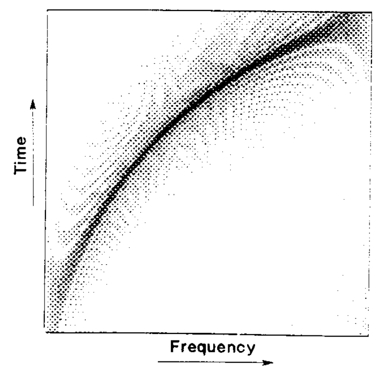

Fig. 1. Affine t-f representation of a hyperbolic signal ($f_{max}/f_{min} = 3$; $BT = 40$)

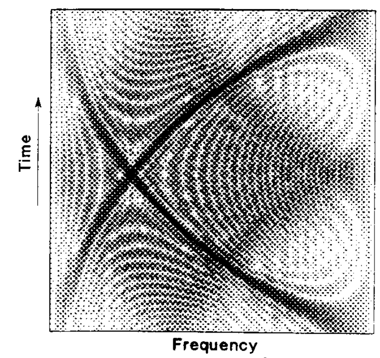

Fig. 3. Affine t-f representation of crossed hyperbolic signals ($f_{max}/f_{min} = 3$; $BT = 40$)

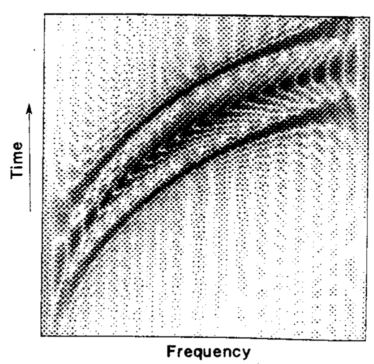

Fig. 2. Affine t-f representation of non-intersecting hyperbolic signals ($f_{max}/f_{min} = 3$; $BT = 40$)

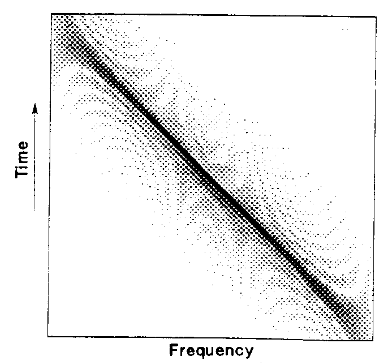

Fig. 4. Affine t-f representation of a chirp ($f_{max}/f_{min} = 1.1$; $BT = 40$)