Evaluation of Radar Detection Performances in Low Grazing Angle Clutter Environment

JAY Emmanuelle †‡, OVARLEZ Jean-Philippe ‡, DECLERCQ David †, DUVAUT Patrick †

†ENSEA/UCP-ETIS, URA D2235 - 6 Avenue du Ponceau, BP 44, 95014 Cergy Pontoise Cedex, France

‡Office National d'Études et de Recherches Aérospatiales ONERA/DEMR/TSI - BP72, F92322 Châtillon Cedex, France

email : jay@onera.fr, ovarlez@onera.fr, declercq@ensea.fr, duvaut@ensea.fr

Tel : +33 (0) 1 69 93 62 96 (Jay) / 69 93 63 13 (Ovarlez) / 30 73 62 97 (Declercq) / 30 73 62 94 (Duvaut)

Abstract : The clutter encountered in low grazing angle situations is generally a non gaussian impulsive noise resulting in a mismatching of the classical radar detector adjusted to the detection threshold of the gaussian hypothesis. To estimate the true detection performances of the radar, one has to take into account not only the power of the noise (as made in the classical "Constant False Alarm Rate " (CFAR) detector) but also its distribution. The method of radar detection performances analysis described in this paper consists, first, in modelisation, thanks to Padé approximation, of the true Probability Density Function (PDF) of the noise envelope (clutter, thermal noise and clutter, Radar Cross Section (RCS) fluctuations, . . .) from experimental data and exploits the special mathematical structure of these estimated PDF in order to, in a second step, evaluate the capability of radar detection of a target (fluctuating or not) which would be embedded, in phase and amplitude, in this noise. This method is also used in order to derive the expressions of the Optima Radar Detectors in estimating the so-called *characteristic function*, characterizing the non-gaussianity of the multidimensionnal clutter in " SIRP " representation (Spherically Invariant Random Process). It is also possible to use the method to evaluate the performances of those detectors.

1 Introduction

Clutter measurements made from experiments by ONERA and other organizations like MIT [5] have shown a strong difference between reality and the standard statistical models used, when the target is moving at very low elevations (with an incidence of less than a few degrees) or with increasing radar range resolution (reducing the number of elementary clutter scatterers). In these situations, the overall clutter statistics can no longer be related to a gaussian's one, but rather to laws characterized by a higher number of degrees of freedom.

To estimate the radar detection performances of a target embedded in grazing angle non gaussian clutter environment, the classical way consists in modeling the probability density function (PDF) of the noise by an *a priori* known law (K-distribution, Weibull, Log-normale, SIRP processes, . . .). Under this hypothesis made on the noise nature, the PDF of target signal (which can be defined or not by its *a priori* known RCS fluctuations law) has to be mathematically determined for evaluating the Radar Operational Curves (ROC) for differents Signal-to-Noise Ratio (SNR) and Probability of False Alarm (Pfa). This kind of procedure described on **figure (1)** depends nevertheless on the statistical *a priori* model of the clutter and does not always lead to a simple or existing mathematical expression (this is for example the case of the mathematical expression of the PDF of a constant signal in Weibull or K-distribution noise).

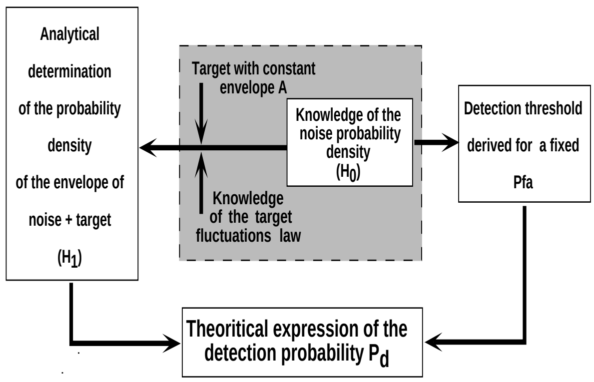

Figure 1: Classical procedure of radar detection performances analysis

The method described in this paper and based on Padé Approximation theory ([2, 3]) allows to estimate the true detection test statistics and so to evaluate the performances of this same test against the present clutter. By this way, either we evaluate the true performances of the test matched to the clutter (for example Optimum Gaussian Detector (OGD) against gaussian clutter) or we show the mismatching of a particular test when the clutter is no more a gaussian one (ex : OGD against K-distributed clut-

ter). We first use the method to evaluate the performances of the envelope detector (similar structure as the OGD) for one pulse of received signal. The envelope of the received signal is calculated (without forming the Likelihood Ratio Test (LRT)) and the expressions of Pfa and Pd (Probability of detection) are mathematically derived.
The other principal use of the method is to estimate the *characteristic function* of the SIRV (Vector) which represents the fluctuation law of the conditionnally gaussian clutter variance. The mathematical expression of this estimated PDF allows to integrate over this PDF in order to derive the general expressions of the joint clutter PDF instead of computing numerical integration ([2, 3]). It reduces the computation.

2 General Relations of the Detection Theory

2.1 Likelihood Ratio Test (LRT)

We consider here the basic problem of detecting the presence or absence of a complex signal **s** in a set of N measurements of m complex vectors $\mathbf{y} = \mathbf{y}_\mathrm{I} + \mathrm{j}\,\mathbf{y}_\mathrm{Q}$ corrupted by a sum of independent additive complex noises **c** corresponding to the clutter echoes and white gaussian thermal noise. It is assumed that the vectors $\mathbf{y}_\mathrm{I}$ and $\mathbf{y}_\mathrm{Q}$, the respectively the in-phase (I) and quadrature (Q) components, are independent and identically distributed (iid) random vectors. The problem can be described mathematically in terms of a hypothesis test between the following pair of statistical hypothesis, where **c** denotes all the unwanted noises :

$$H_0 : \mathbf{y} \;=\; \mathbf{c} \qquad (1)$$
$$H_1 : \mathbf{y} \;=\; \mathbf{s} + \mathbf{c} \qquad (2)$$

When the target signal **s** is present it corresponds to a modified version of the perfectly known transmitted signal **p**, that is to say that **s** can be rewritten as $\mathbf{s} = T(A, \underline{\theta})\,\mathbf{p}$. We denote by A the target amplitude and we suppose determined all the others parameters ($\underline{\theta}$) which characterize the target (Doppler frequency, time delay, ...).

The observed vector **y** is used to form the likelihood ratio $\Lambda(\mathbf{y})$ which is compared with a threshold η set to a desired P_{fa} value :

$$\Lambda(\mathbf{y}) = \frac{p_{\mathbf{y}}(\mathbf{y}/H_1)}{p_{\mathbf{y}}(\mathbf{y}/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \qquad (3)$$

The likelihood ratio so formed is data depending ; the resulting detectors structures are also data depending and their associated performances follow from the statistic of the data. The false alarm probability P_{fa} is the probability of choosing H_1 when the target is absent, i.e. :

$$P_{fa} = \mathbb{P}(\Lambda(\mathbf{y}) \underset{H_0}{>} \eta), \qquad (4)$$

and the detection probability is the probability of choosing H_1 when the target is present, i.e. :

$$P_d = \mathbb{P}(\Lambda(\mathbf{y}) \overset{H_1}{>} \eta). \qquad (5)$$

The Neymann-Pearson criterion consists in fixing P_{fa} while maximizing P_d.

2.2 Gaussian clutter case

When the clutter **c** is supposed to be complex gaussian distributed with zero mean, variance $2\sigma^2$ and covariance matrix $2\sigma^2\mathbf{M}$ ($\mathcal{CN}(0, 2\sigma^2\mathbf{M})$) we have :

$$p_{\mathbf{y}}(\mathbf{y}/H_0) \;=\; \frac{1}{(\pi\, 2\,\sigma^2)^m |\mathbf{M}|} \exp\left(-\frac{\mathbf{y}^\dagger\,\mathbf{M}^{-1}\mathbf{y}}{2\,\sigma^2}\right) \qquad (6)$$
$$p_{\mathbf{y}}(\mathbf{y}/H_1) \;=\; p_{\mathbf{y}}(\mathbf{y} - \mathbf{s}/H_0) \qquad (7)$$

and the likelihood ratio can be rewritten as :

$$-(\mathbf{y}-\mathbf{s})^\dagger\,\mathbf{M}^{-1}(\mathbf{y}-\mathbf{s}) + \mathbf{y}^\dagger\,\mathbf{M}^{-1}\mathbf{y} \underset{H_0}{\overset{H_1}{\gtrless}} 2\,\sigma^2\lambda, \qquad (8)$$

where $\lambda = \log(\eta)$. Most of the time, the target signal **s** is unknown and an estimate in the Maximum Likelihood sense (ML) of the non fluctuating target amplitude A is derived ($\mathbf{s} = A\mathbf{p}$) :

$$\hat{A}_{ML} = \underset{A}{\operatorname{argmax}}\,\Lambda(\mathbf{y}) = \frac{\mathbf{p}^\dagger\,\mathbf{M}^{-1}\,\mathbf{y}}{\mathbf{p}^\dagger\,\mathbf{M}^{-1}\,\mathbf{p}} \qquad (9)$$

This estimate stays valid in the rest of this paper because of the SIRV representation which keeps a gaussian form under the integral and maximizing the resulting LRT over A is always maximizing (8).
The resulting so-called Optimum Gaussian Detector (OGD) comes from the Generalized Likelihood Ratio Test (GLRT) and is :

$$|\mathbf{p}^\dagger\,\mathbf{M}^{-1}\,\mathbf{y}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2\lambda\,\mathbf{p}^\dagger\mathbf{M}^{-1}\mathbf{p} \qquad (10)$$

The OGD detector just compares the matched filter output to the threshold. Using this quadratique test is equivalent to use an envelope detector structure with an adjusted threshold.
Given that **y** is gaussian distributed, the laws under H_0 and H_1 of this detection strategies are respectively exponential and Rice-Nagakami distributed and we can derive the expressions of P_{fa} and P_d :

$$P_{fa} \;=\; e^{-\lambda} \qquad (11)$$
$$P_d \;=\; \mathcal{Q}\left(\sqrt{\frac{A^2}{\sigma^2\,\mathbf{p}^\dagger\mathbf{M}^{-1}\mathbf{p}}}, \sqrt{\frac{\lambda}{\sigma^2\,\mathbf{p}^\dagger\mathbf{M}^{-1}\mathbf{p}}}\right), \qquad (12)$$

where

$$\mathcal{Q}(a, b) = \int_b^{+\infty} x\,\exp\left(-\frac{x^2 + a^2}{2}\right)\, I_0(ax)dx \qquad (13)$$

is the Marcum $\mathcal{Q}$-function. If the amplitude of the target is fluctuating among $p(A;A_0)$ the expression of P_d has to be integrated over $p(A;A_0)$. For example we can consider the Swerling-K fluctuations for the target amplitude A (with $\mathbb{E}(A)=A_0^2$) given by :

$$p(A;A_0)=\frac{2}{\Gamma(K)}\left(\frac{K}{A_0^2}\right)^K A^{2K-1}\exp\left(-\frac{KA^2}{A_0^2}\right) \tag{14}$$

where K is the parameter of the Swerling fluctuation. Therefore, integrating with respect to $p(A;A_0)$ yields to the expression of P_d for a Swerling-K fluctuating target ([1]) :

$$\begin{aligned} P_d &= 1-(1+Z)^{-K} \\ &\times \int_0^T x\,\exp(-\frac{x^2}{2})M\left(K,1;\frac{Z\,x^2}{1+Z}\right)\,dx \\ T &= \sqrt{\frac{\lambda}{\sigma^2\mathbf{p}^\dagger\mathbf{M}^{-1}\mathbf{p}}} \\ Z &= \frac{A_0^2}{K\sigma^2\mathbf{p}^\dagger\mathbf{M}^{-1}\mathbf{p}} \end{aligned} \tag{15}$$

where $M(a,b;x)$ is the confluent hypergeometric function (an alternative notation is ${}_1F_1(a,b;x)$) with parameters a, b, and argument x. The calculation of $M(K,1;x)$ (K is an integer) is obtained by a recurrence relation given $\forall n\geq 1$ by ([1]) :

$$\begin{aligned} M(1,1;x) &= e^x \\ M(2,1;x) &= (1+x)\,e^x \\ M(n+2,1;x) &= \frac{1}{n+1}\left[(2n+1+x)\,M(n+1,1;x)\right. \\ &\quad \left.-(n+1)\,M(n,1,x)\right] \end{aligned} \tag{16}$$

For a special K value (example $K=1$ gives the Swerling-I law) the computation of the P_d expression is not too heavy because of the definite integral and the recurrence relation (16).

2.3 Non-Gaussian clutter case

In the case of non-gaussian clutter, the detection strategies can be derived if we consider a particular clutter nature, i.e. if an *a priori* hypothesis is made on the clutter statistic. On the other hand the expressions of P_{fa} and P_d are rather impossible to derive analytically.
We propose a method to solve this latter point in estimating the PDF of the test, thanks to Padé approximation ([2, 3]), and we first study a particular detector that is equivalent to the gaussian one, the envelope detector, that we call *RSOGD* for Root-Squared OGD. In this way, we evaluate the one-pulse performances of the OGD when the disturbances are non-gaussian. Before we describe the method of Padé approximation.

3 The Padé approximation method

Given a random variate Z with PDF $p(z)$ the Moment Generating Function (MGF) of this variate is defined by :

$$\Phi(u)=\int_0^{+\infty}p(z)\,e^{-uz}\,dz=\sum_{n\geq 0}\frac{\mu_n\,(-u)^n}{n!}=\sum_{n\geq 0}c_n\,u^n, \tag{17}$$

where μ_n denotes the n-order moment of Z.

If we suppose all the moments μ_n perfectly known up to order $L+M+1$, the main idea is to truncate the infinite series at the order $L+M+1$ and to approximate it by a rational function $P^{[L/M]}(u)$ ($L\leq M$) defined by :

$$P^{[L/M]}(u)=\frac{\sum_{n=0}^{L}a_n\,u^n}{\sum_{n=0}^{M}b_n\,u^n}, \tag{18}$$

where the coefficients $\{a_n\}$ et $\{b_n\}$ are determined so that the following equality be verified :

$$\frac{\sum_{n=0}^{L}a_n\,u^n}{\sum_{n=0}^{M}b_n\,u^n}=\sum_{n=0}^{L+M}c_n\,u^n+\mathcal{O}(u^{L+M+1}). \tag{19}$$

The notation $\mathcal{O}(u^{L+M+1})$ simply takes into account terms of order higher than u^{L+M}. To determine the two sets of coefficients $\{a_n\}$ and $\{b_n\}$, we have to match the coefficients :

$$\sum_{n=0}^{M}b_n\,u^n\sum_{n=0}^{L+M}c_n\,u^n = \sum_{n=0}^{L}a_n\,u^n+\mathcal{O}(u^{L+M+1}). \tag{20}$$

The moments matching conditions fix in a first step the set of coefficients $\{b_n\}$ by solving a simple set of M linear equations for the M unknown denominator coefficients :

$$\sum_{n=0}^{M}b_n\,c_{L-n+j}=0,\qquad 1\leq j\leq M. \tag{21}$$

In a second step the set $\{a_n\}$ is determined by a simple convolution of the $\{b_n\}$ and the $\{c_n\}$ coefficients :

$$a_j=c_j+\sum_{i=1}^{\min(M,j)}b_i\,c_{j-i},\qquad 0\leq j\leq L. \tag{22}$$

The set of coefficients $\{a_n\}$ and $\{b_n\}$ determined with (21) and (22), defines, thanks to the Padé Approximation,

the *One Point* parametric modeling of the MGF given its power series expansion (17) about $u = 0$.
If we suppose that the rational fraction approximation has M distinct poles with negative real part to assume its convergence for $u \to \infty$, the relation (18) can be rewritten as :

$$P^{[L/M]}(u) = \sum_{k=1}^{M} \frac{\lambda_k}{u + \alpha_k} \qquad \mathrm{Re}(\alpha_k) > 0. \tag{23}$$

From this description, we are able to determine a random vector's PDF and CDF using the Inverse Laplace Transform of the corresponding MGF performed by residue inversion formula leading to a sum of weighted decaying exponentials :

$$\tilde{p}(z) = \sum_{k=1}^{M} \lambda_k \, e^{-\alpha_k \, z} \tag{24}$$

$$\tilde{F}(z) = 1 - \sum_{k=1}^{M} \frac{\lambda_k}{\alpha_k} \, e^{-\alpha_k \, z} \tag{25}$$

The coefficients $\{\lambda_k\}_{k=1:M}$ and $\{\alpha_k\}_{k=1:M}$ are complex, all in pairs conjugate if M is even and in pairs conjugate except an odd number of them if M is odd.

The first application given to this method is to evaluate the PDFs of the envelope of one pulse of the received signal. From one pulse the statistics of the test comes directly from the envelope of the data. It is just necessary to estimate its PDF. From the whole received vector and to compute the likelihood ratio, it is necessary to know or to estimate the joint PDF of the vector what is not possible to do with a Padé approximation. In this latter case we will interest to estimate the PDF of the detection tests which are one-dimensionnal positive random variables.

4 One pulse RSOGD performances

4.1 RSOGD or envelope detector

For one pulse, the envelope detector principle is to evaluate the envelope of the received signal $y(t)$. We denote by $p_{H_0}(r)$ and $p_{H_1}(r)$ the probability density functions (PDF) of the envelope of the received signal $y(t)$ respectively under H_0 and H_1 hypothesis. The P_{fa} and P_d values are so defined by :

$$P_{fa} = \mathbb{P}(|y(t)| \underset{H_0}{>} \theta) = \int_{\theta}^{+\infty} p_{H_0}(r)\, dr \tag{26}$$

$$P_d = \mathbb{P}(|y(t)| \overset{H_1}{>} \theta) = \int_{\theta}^{+\infty} p_{H_1}(r)\, dr \tag{27}$$

The expression of $p_{H_1}(r)$ can be directly deducted from $p_{H_0}(r)$ if consider their respective *radial coherent characteristic functions*. The detail of the calculation can be found in ([10, 8]) and the resulting expression is :

$$\begin{aligned} p_{H_1}(r; A_0) &= \int_0^{+\infty} p_{H_1}(r; A)\, p(A; A_0)\, dA \\ &= \int_0^{+\infty}\int_0^{+\infty} r\rho\, J_0(\rho r)\, J_0(\rho y)\, p_{H_0}(y) \\ &\times \left[\int_0^{+\infty} J_0(\rho A) p(A; A_0) dA\right] d\rho dy \end{aligned} \tag{28}$$

where $p(A; A_0)$ is the fluctuations law of the target ($p(A; A_0) = \delta(A - A_0)$ in the case of non-fluctuating target, A_0 being the mean level of the fluctuations). This expression is very interesting and the whole problem is related in the determination or estimation of the PDF of the only noise.

4.2 Performances evaluation for one pulse

The latter remark holds in the problem of evaluating the performances of the detector. From (26) and (27) we can see that the expressions of P_{fa} and P_d are given by the statistics of the test under H_0 and H_1. From the envelope of the received data we can estimate these statistics thanks to Padé approximation. We consider then that :

$$p_{H_0}(r) = \sum_{k=1}^{M} \lambda_k \, e^{-\alpha_k \, r}. \tag{29}$$

With simple calculation P_{fa} expression (26) becomes :

$$P_{fa} = \sum_{k=1}^{M} \frac{\lambda_k}{\alpha_k} \, e^{-\alpha_k \, \theta}, \tag{30}$$

and the detection threshold θ is obtained with a desired P_{fa} value by the determination of this equation. Using (5) and (28) a general P_d expression is derived for fluctuating or not target (the same notation is kept for $p(A; A_0)$) :

$$\begin{aligned} P_d &= 1 - \int_0^{+\infty} \theta J_1(\rho\theta) \sum_{k=1}^{M} \frac{\lambda_k}{\sqrt{\rho^2 + \alpha_k^2}} \\ &\times \int_0^{+\infty} J_0(\rho A) p(A; A_0) dA d\rho. \end{aligned} \tag{31}$$

In the same way it is possible to estimate $p(A; A_0)$ by Padé approximation (P coefficients $\{\gamma_i\}$ and $\{\delta_i\}$) :

$$\tilde{p}(A; A_0) = \sum_{i=1}^{P} \frac{\gamma_i}{A_0} e^{-\frac{\delta_i}{A_0} A}, \tag{32}$$

and the P_d expression is, after few calculation :

$$\begin{aligned} P_d &= 1 - \int_0^{+\infty} \theta J_1(\rho\theta) \sum_{k=1}^{M} \frac{\lambda_k}{\sqrt{\rho^2 + \alpha_k^2}} \\ &\times \sum_{i=1}^{P} \frac{\gamma_i}{\sqrt{\rho^2 A_0^2 + \delta_i^2}} d\rho. \end{aligned} \tag{33}$$

In that way we are able to determine the performances of the RSOGD for one received signal pulse whatever the nature the present clutter (often non-gaussian one). In the next section we evaluate the performances of the OGD detector for the m-train of pulses thanks to the Padé approximation.

4.3 m pulses OGD performances estimated with Padé

We have seen that the P_{fa} and P_d expressions come from the statistics of the test (26,27). In the case of the OGD detector, these expressions are known if the noise is gaussian or if we know its PDF. With Padé approximation, we are able to evaluate the OGD performances whatever the statistic of the noise is. The OGD detector (10) is for the m pulses :

$$|\mathbf{p}^\dagger \mathbf{M}^{-1} \mathbf{y}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2 \lambda \, \mathbf{p}^\dagger \mathbf{M}^{-1} \mathbf{p} = \phi$$

The left term is a random variate V_i (data depending) with PDF $p(V_i)$. $i = 0$ for H_0 hypothesis and $i = 1$ for H_1 hypothesis. So :

$$\tilde{p}(V_i) = \sum_{k=1}^{M} \lambda_{k,i} \, e^{-\alpha_{k,i} \, V_i}, \tag{34}$$

and

$$\begin{aligned} P_{fa} &= \int_{\phi}^{+\infty} \tilde{p}(V_0) dV_0 \\ &= \sum_{k=1}^{M} \frac{\lambda_{k,0}}{\alpha_{k,0}} \exp(-\alpha_{k,0} \, \phi) \end{aligned} \tag{35}$$

$$\begin{aligned} P_{d} &= \int_{\phi}^{+\infty} \tilde{p}(V_1) dV_1 \\ &= \sum_{k=1}^{M} \frac{\lambda_{k,1}}{\alpha_{k,1}} \exp(-\alpha_{k,1} \, \phi) \end{aligned} \tag{36}$$

This method allows to treat directly with the m-train of the received data pulses which are used in the detector structure.

To solve the problem of modelization of the non-gaussianity of the clutter, the clutter process can be modeled as a gaussian process with a variance which is itself a random variate. This model results in the so-called Spherically Invariant Random Process (SIRP) and is explained, for example, in ([13, 11]). Many people exploited this representation to apply it in radar detection. In ([7]) are derived the optimum radar receivers to detect fluctuating and non-fluctuating targets against a mixture of K-distributed and gaussian clutter with perfectly known statistics.
In the next section we describe briefly the SIRP theory and the results found in ([7, 4]). Then, with always the idea of having non *a priori* assumption on the clutter statistic, we present similar results after estimating the PDF of the gaussian process variance with a Padé approximation. This may be applied after a *Maximum A Posteriori* estimation of the variance of the conditionnally gaussian process (the process is conditionnally to the variance a gaussian one). The N estimates are then considered as a N-sample of a positive random variate ; a Padé approximation is used to derive the estimated PDF of the variances of the N observations of the clutter and so characterize the non-gaussianity of the clutter without *a priori*. Once the estimate optimum detector (EOD) is derived it is possible to evaluate its performances using a Padé approximation.

5 Contributions of Padé approximation to the Optimum Radar Detector

5.1 SIRP - Description

When the clutter is non-gaussian we use the SIRP representation that consider the clutter process as the product of a m complex gaussian vector $\mathbf{x} = \mathbf{x}_I + j\mathbf{x}_Q$ and a positive random variate τ :

$$\mathbf{c} = \mathbf{x}\sqrt{\tau}. \tag{37}$$

The in-phase ($\mathbf{x}_I$) and quadrature ($\mathbf{x}_Q$) components are independent and identically distributed random vectors with zero-mean, unit variance and covariance matrix $\mathbf{M}$. The vector $\mathbf{x}$ is then zero-mean with variance 2 and covariance matrix $2\mathbf{M}$. The PDF of the variable τ is the so-called *characteristic function* of the SIRP and the so formed vector $\mathbf{c}$ is, conditionnally to τ, a complex gaussian random process with zero-mean, variance 2τ and covariance matrix $2\tau\mathbf{M}$:

$$\begin{aligned} p(\mathbf{c}/\tau) &= \frac{1}{\pi^m \, |2\tau\mathbf{M}|} \exp\left(-\mathbf{c}^\dagger (2\tau\mathbf{M})^{-1}\mathbf{c}\right) \\ &= \frac{1}{(2\pi\tau)^m \, |\mathbf{M}|} \exp\left(-\frac{\mathbf{c}^\dagger\mathbf{M}^{-1}\mathbf{c}}{2\tau}\right) \end{aligned} \tag{38}$$

The PDF of the vector $\mathbf{c}$ is derived after integration over $p(\tau)$:

$$p(\mathbf{c}) = \int_0^{+\infty} \frac{1}{(2\pi\tau)^m |\mathbf{M}|} \exp\left(-\frac{\mathbf{c}^\dagger\mathbf{M}^{-1}\mathbf{c}}{2\tau}\right) p(\tau) d\tau \tag{39}$$

5.2 SIRP - Optimum Radar Detector

Applied to the detection criterion, the latter expression is in fact $p_{\mathbf{y}}(\mathbf{y}/H_0)$ and $p_{\mathbf{y}}(\mathbf{y}/H_1) = p_{\mathbf{y}}(\mathbf{y} - \mathbf{s}/H_0)$. The likelihood ratio becomes ([7]) :

$$\int_0^{+\infty} \left[\exp\left(-\frac{q_1(\mathbf{y})}{2\tau}\right) - \exp\left(\lambda - \frac{q_0(\mathbf{y})}{2\tau}\right)\right] \frac{p(\tau)}{\tau^m} d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0 \tag{40}$$

where $q_0(\mathbf{y}) = \mathbf{y}^\dagger\mathbf{M}^{-1}\mathbf{y}$, $q_1(\mathbf{y}) = q_0(\mathbf{y} - \mathbf{s})$ and $\lambda = \ln(\eta)$. As before, the amplitude A of the target ($\mathbf{s} = A\mathbf{p}$) is unknown but estimated in the *Maximum Likelihood* sense (cf. (9)). In this case the detection strategy is given by (40) where now :

$$q_1(\mathbf{y}) = \mathbf{y}^\dagger \mathbf{M}^{-1}\mathbf{y} - \frac{|\mathbf{p}^\dagger \mathbf{M}^{-1}\mathbf{y}|^2}{\mathbf{p}^\dagger \mathbf{M}^{-1}\mathbf{p}} \qquad (41)$$

5.3 SIRP - Example : Optimum K Detector : the OKD

Given an expression for $p(\tau)$ the Optimum Detector is obtained for the so assumed clutter statistic and after computation of the generalized integration over τ. In the case of K-distributed clutter (size m) with parameters ν and b (ν is the form parameter whose value determine the spikiness of the distribution, $q(\mathbf{x}) = \mathbf{x}^\dagger\, \mathbf{M}_X^{-1}\mathbf{x}$) :

$$p(\mathbf{x}) = \frac{2\, b^{\nu+m}}{\pi^m\, |\mathbf{M}_X| \Gamma(\nu)\, 2^{\nu+m}}\, q(\mathbf{x})^{\frac{\nu-m}{2}}\, K_{\nu-m}(b\,\sqrt{q(\mathbf{x})}), \qquad (42)$$

the PDF of τ is a Gamma(ν,$\beta = 2/b^2$) PDF whose expression is :

$$p(\tau) = \frac{\tau^{\nu-1}}{\Gamma(\nu)\,\beta^\nu}\, \exp\left(-\frac{\tau}{\beta}\right). \qquad (43)$$

Integrating (40) with respect to this PDF gives the expression of the so-called Optimum K-distributed Detector (OKD) $\forall m \geq 2$:

$$\left(\frac{q_1(\mathbf{y})}{q_0(\mathbf{y})}\right)^{\dfrac{\nu-m}{2}} . \frac{K_{\nu-m}\left(b\,\sqrt{q_1(\mathbf{y})}\right)}{K_{\nu-m}\left(b\,\sqrt{q_0(\mathbf{y})}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \qquad (44)$$

If $m = 1$, $q_1(\mathbf{y}) = 0$ and the equation (40) becomes:

$$(q_0(\mathbf{y}))^{\frac{\nu-1}{2}}\, K_{\nu-1}\left(b\,\sqrt{q_0(\mathbf{y})}\right) \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{(2\,\beta)^{\frac{\nu-1}{2}}\,\Gamma(\nu)}{2\,\nu\,\eta} \qquad (45)$$

5.4 Padé Estimated Optimum Detector : the PEOD

Recalling that the process is gaussian if considered it conditionnally to τ, it is possible to estimate the variance τ for each observation vector (N observations). This can be realized either in the ML sense or, to give some more information about the estimates (positive variate, variance of a gaussian vector), in the MAP sense that consists in adding an *a priori* information and to reduce the estimates region to the more realistic one. In this case, the *a priori* information is a conjugate (informative) prior, i.e., the prior combined with the likelihood yield to a posterior density having the same form as the prior density. This method comes from the Bayes'rule :

$$p_{\mathbf{y}}(\tau/\mathbf{y}) \propto p_{\mathbf{y}}(\mathbf{y}/\tau)\, g(\tau) \qquad (46)$$

$p_{\mathbf{y}}(\mathbf{y}/\tau)$ is the likelihood of the data (the conditionnal gaussian density) and $g(\tau)$ is the *prior* or the *a priori* information [12, 9]. We choose $g(\tau)$ as being an Inverse Gamma density with parameters a and d and the estimates are derived as follow :

$$\begin{aligned} g(\tau) &= \frac{\tau^{-a-1}}{d^a\,\Gamma(a)}\, \exp\left(-\frac{1}{d\,\tau}\right) \\ \hat{\tau}_{MAP} &= \underset{\tau}{\operatorname{argmax}}\, p_{\mathbf{y}}(\tau/\mathbf{y}) = \underset{\tau}{\operatorname{argmax}}\, p_{\mathbf{y}}(\mathbf{y}/\tau)\, g(\tau) \\ &= \frac{d\,\mathbf{y}^\dagger\,\mathbf{M}^{-1}\,\mathbf{y} + 2}{2\,d\,(m+a+1)} \end{aligned} \qquad (47)$$

These N *asymptotically unbiaised* estimates represent a N sample of the positive random variance τ. The representative PDF $\tilde{p}(\hat{\tau})$ can be estimated with a Padé approximation :

$$\tilde{p}(\hat{\tau}) = \sum_{k=1}^{M} \lambda_k\, e^{-\alpha_k\,\hat{\tau}}, \qquad (48)$$

and the calculation of (39) if replaced in the detection case is tractable ([6]) to give :

$$\begin{aligned} p_{\mathbf{y}}(\mathbf{y}/H_0) &= \frac{1}{(2\pi)^m |\mathbf{M}|} \sum_{k=1}^{M} \lambda_k \\ &\times \int_0^{+\infty} \tau^{-m} \exp\left(-\frac{\mathbf{y}^\dagger \mathbf{M}^{-1}\mathbf{y}}{2\tau} - \alpha_k \tau\right) d\tau \\ &= \frac{2^{(1-m)/2}}{\pi^m |\mathbf{M}|} \sum_{k=1}^{M} \lambda_k \left(\frac{q_0(\mathbf{y})}{\alpha_k}\right)^{(1-m)/2} \\ &\times K_{1-m}\left(\sqrt{2\,q_0(\mathbf{y})\,\alpha_k}\right) \end{aligned} \qquad (49)$$

The likelihood ratio compared with the threshold η (3) becomes :

$$\frac{p_{\mathbf{y}}(\mathbf{y}/H_0)}{p_{\mathbf{y}}(\mathbf{y}-\mathbf{s}/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \qquad (50)$$

and finally we have $\forall m \geq 2$:

$$\left(\frac{q_1(\mathbf{y})}{q_0(\mathbf{y})}\right)^{\dfrac{1-m}{2}} . \frac{\sum_{k=1}^{M} \lambda_k (\alpha_k)^{\frac{m-1}{2}} K_{1-m}\left(\sqrt{B_k^1(\mathbf{y})}\right)}{\sum_{k=1}^{M} \lambda_k (\alpha_k)^{\frac{m-1}{2}} K_{1-m}\left(\sqrt{B_k^0(\mathbf{y})}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \qquad (51)$$

where

$$\begin{aligned} q_0(\mathbf{y}) &= \mathbf{y}^\dagger\,\mathbf{M}^{-1}\,\mathbf{y} \\ q_1(\mathbf{y}) &= q_0(\mathbf{y}-\mathbf{s}) \\ & \quad \mathbf{s} \text{ perfectly known} \\ q_1(\mathbf{y}) &= \mathbf{y}^\dagger \mathbf{M}^{-1}\mathbf{y} - \frac{|\mathbf{p}^\dagger \mathbf{M}^{-1}\mathbf{y}|^2}{\mathbf{p}^\dagger \mathbf{M}^{-1}\mathbf{p}} \\ & \quad \mathbf{s} \text{ unknown} \\ B_k^1(\mathbf{y}) &= 2\,\alpha_k q_1(\mathbf{y}) \\ B_k^0(\mathbf{y}) &= 2\,\alpha_k q_0(\mathbf{y}) \end{aligned}$$

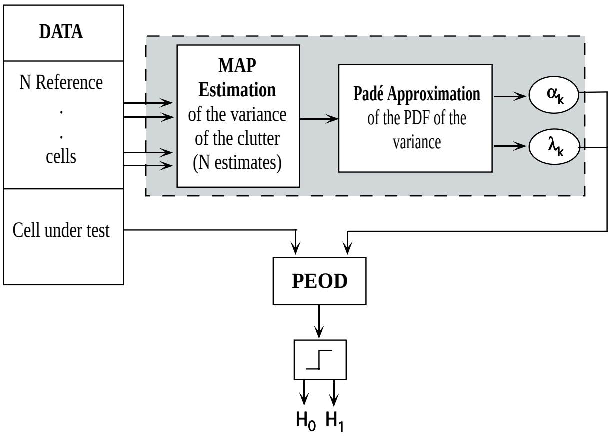

Figure 2: Block diagram for computation of the PEOD.

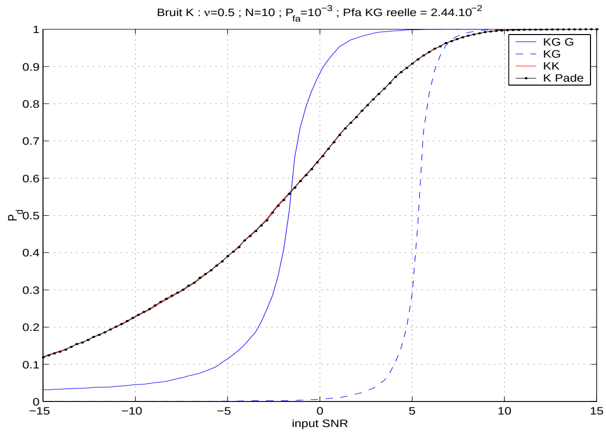

Figure 3: Performances comparison between the OGD, OKD and PEOD for K-distributed clutter ($\nu = 0.5$, $P_{fa} = 10^{-3}$, $m = 10$).

This expression (51) gives a Padé Estimated Optimum Detector (PEOD) when the target is embedded in a clutter modelized as a SIRV. The advantage of Padé approximation is to deal with the data without *a priori* on the clutter statistics.

It is now possible to evaluate the PEOD performances thanks to Padé approximation in the case where the quantity on the left of the expression (51) is positive. This condition is verified because of the following reasons :

- $q_1(\mathbf{y})$ and $q_0(\mathbf{y})$ are two quadratic positives forms,
- Padé coefficients are all in pairs conjugate (except few of them which are reals, among the parity of M),
- $K(\bar{z}) = \overline{K(z)}$,
- each discrete sum in the quotient is real,
- the two discrete sums have the same sign.

On **figure (2)** is represented a block diagram which summarizes the approach of PEOD. The PEOD structure depends only on Padé coefficients calculated from a series of N reference clutter cells. Then, with the received data (the cells under test), we just have to compute the PEOD given by (51) using these coefficients and to decide if a target is present or not.

In the next section the results of simulations of the differents approaches are shown. We can see the comparison between the performances of the OGD and PEOD. We also present results of the one pulse RSOGD on experimental forest clutter that we compare with the OGD. We can see that if we suppose to be in a gaussian situation of clutter the P_{fa} value increases.

6 Simulations

In all the simulations we consider an uncorrelated clutter, i.e. the correlation matrix $\mathbf{M}$ is diagonal and supposed to be determined.

We compare the performances of the OGD with those of the OKD (Optimum K Detector) and the PEOD. To represent non-gaussian clutter we choose a K-distributed clutter with differents values of the shape parameter $\nu = 0.5, 1, 2, 20$. When $\nu \longrightarrow +\infty$ the K PDF tends to a gaussian one. More the value of ν increases and closer are the curves plotted for the OKD and PEOD (we can see the evolution with respect to ν value on the series of **figures (3, 4, 5, 6)**).

All the curves represent the detection probability P_d versus the Signal-to-Noise-Ratio given for one pulse. Given that $m = 10$ pulses are considered, the total SNR for the entire pulse is in fact of 10 dB more than for one pulse.
For example, on figures, SNR$= 0$ dB for one pulse corresponds to SNR$= 10$ dB for the 10 coherently integrated pulses.
The plots of P_d versus the *one pulse* SNR are shown on **figures (3, 4, 5, 6)** ($M = 6$ for the Padé approximation). On the last one the PEOD curves (where $\nu = 20$) blends with the OKD curves. We also denote by :

- ”KG G” : K-distributed clutter - OGD - detection threshold derived if suppose Gaussian clutter for a fixed P_{fa} (mismatched value),
- ”KG” : K-distributed clutter - OGD - detection threshold derived by Monte Carlo method with the OGD output of the K clutter for a fixed P_{fa} (true value),
- ”KK” : K-distributed clutter - OKD - detection threshold derived by Monte Carlo method with the OKD output of the K clutter for a fixed P_{fa} (true value),
- ”K Pade” : K-distributed clutter - PEOD - detection threshold derived by Monte Carlo method with the PEOD output of the K clutter for a fixed P_{fa} (true value),

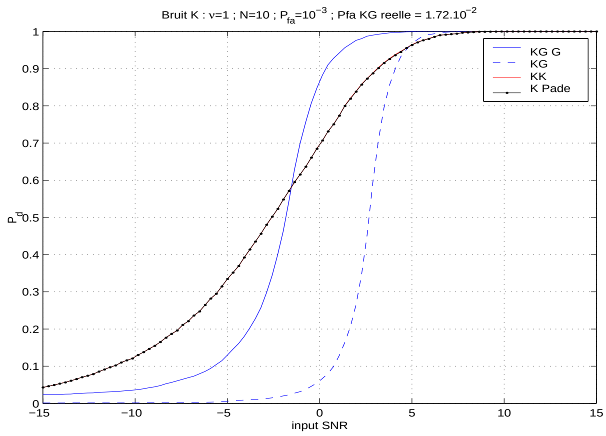

Figure 4: Performances comparison between the OGD, OKD and PEOD for K-distributed clutter ($\nu = 1$, $P_{fa} = 10^{-3}$, $m = 10$).

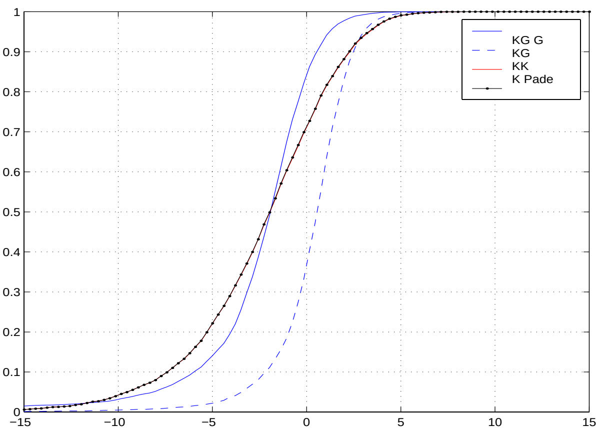

Figure 5: Performances comparison between the OGD, OKD and PEOD for K-distributed clutter ($\nu = 2$, $P_{fa} = 10^{-3}$, $m = 10$)

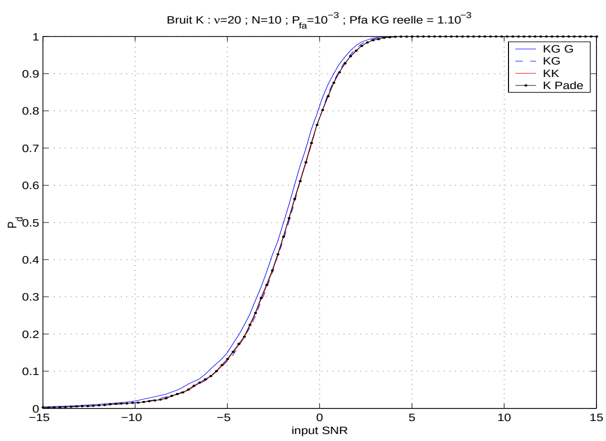

Figure 6: Performances comparison between the OGD, OKD and PEOD for K-distributed clutter ($\nu = 20$, $P_{fa} = 10^{-3}$, $m = 10$)

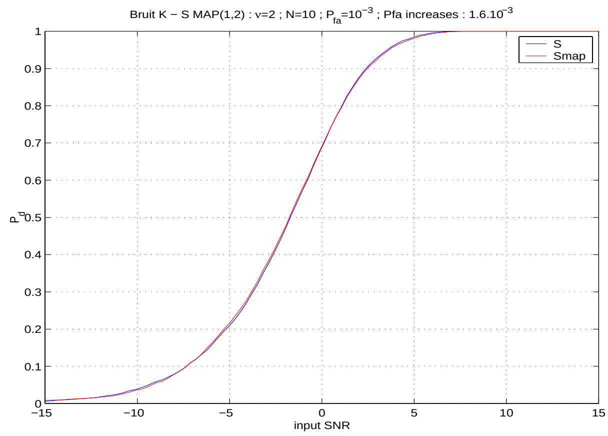

Figure 7: Comparison for the PEOD with K-distributed clutter ($\nu = 2$, $P_{fa} = 10^{-3}$, $m = 10$) between the PDF of τ just estimated by Padé and the PDF estimated by Padé after MAP Estimation

In the case of " KG G " we calculate the " true P_{fa} KG " which would be needed to obtain such threshold value (lower than the true one because of the Gaussian hypothesis). We confirm the increasing of false alarm (by a factor ten) if gaussian hypothesis is made on the clutter statistic.

On **figure (7)** we study the SNR loss when the PDF of the variance of the conditionnally gaussian clutter is estimated with Padé approximation after MAP estimation with (47).

The histograms of the true τ samples and of the MAP *asymptotically unbiaised* estimated samples $\hat{\tau}_{map}$ are shown on **figure (8)** for a K-distributed clutter ($\nu = 2$). We note that the tiny loss does not really perturb the detection capability.

On **figure (9)** we have plotted the performances of the PEOD after MAP estimation of τ (the histogram of the estimates is on **figure (10)**) if consider a Weibull clutter whose *characteristic function* (SIRV representation) is unknown as well as the analytical expressions of P_d and P_{fa}.

We also apply the method on experimental forest clutter. The data are clutter data only, and to derive the SNR/P_d curves we consider that a virtual non fluctuating unknown target is embedded in.

We show the mismatch of the gaussian hypothesis by a high increase of false alarm in this case. The detector is the RSOGD and the curves are shown on **figures (11, 12)**.

7 Conclusions and outlooks

The present paper has addressed the contribution of Padé approximation method in the problem of coherent radar detection of an unknown target amplitude statistics.

The detection strategy has been derived by calculating the likelihood ratio test assuming that the non gaussian clutter is a SIRV. The *characteristic function* PDF of the

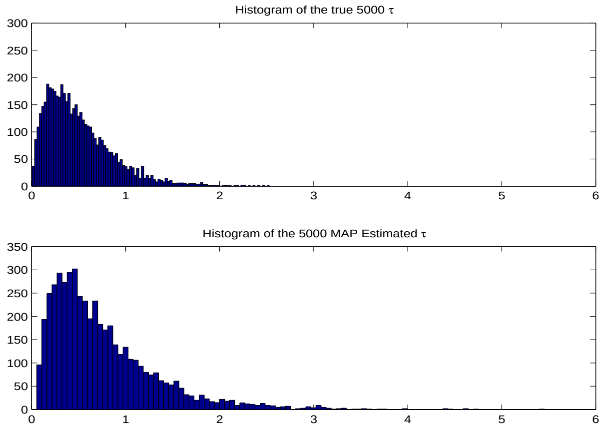

Figure 8: Histograms of the true samples of τ and of the MAP estimated samples $\hat{\tau}_{map}$ in relation with figure (7).

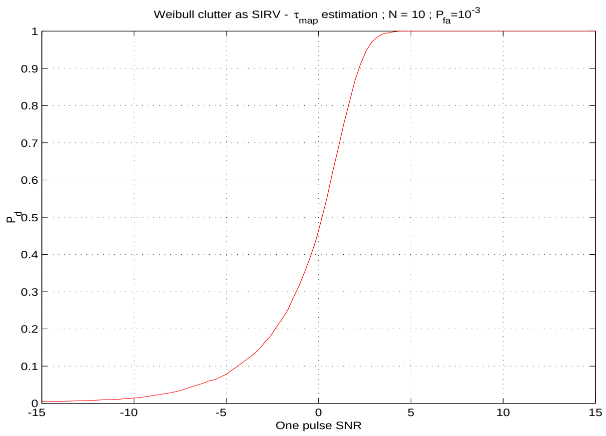

Figure 9: PEOD with Weibull distributed clutter ($a = 0.2$, $b = 2$; $P_{fa} = 10^{-3}$, $m = 10$). The PDF of τ is unknown for Weibull PDF : it is estimated by Padé after a MAP estimation of the N τ values if consider a SIRV representation.

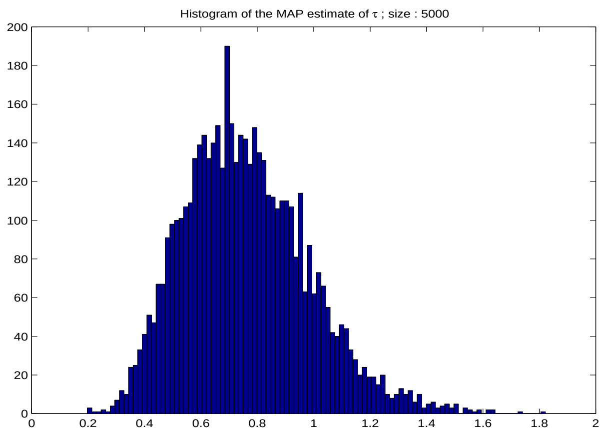

Figure 10: Histogram of the 5000τ MAP estimated for the Weibull clutter used in figure (9)

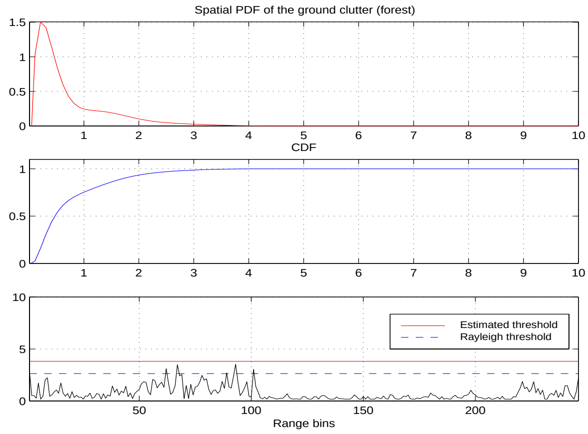

Figure 11: Approximation of the envelope of forest clutter PDF from experimental data

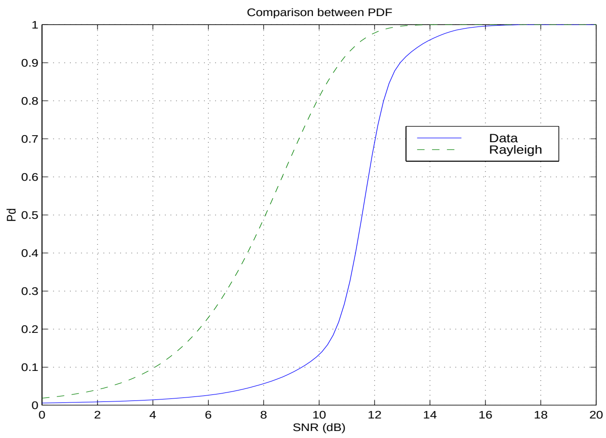

Figure 12: Comparison between radar detection performances of non fluctuating target virtually embedded in the forest clutter (figure (11)) and this same target in the same clutter supposed gaussian. $Pfa = 10^{-3}$ gives a threshold of 3.23 for the forest clutter and 2.63 if gaussian clutter ; this latter value would correspond to a $P_{fa} = 1.626.10^{-2}$ for the forest clutter

SIRV has been estimated thanks to Padé approximation and it allows to analytically derive the expression of the Optimum Detector for any clutter statistic.
Another important point is that modelizing the non gaussian clutter with a SIRP representation says that, observation-by-observation, the clutter is gaussian conditionnally to its variance which determines the nature of the clutter. It is possible to estimate the variance observation-by-observation and then to use a Padé approximation for the PDF of this estimated sample.
So, the expression of the Optimum Detector (called PEOD for Pade Estimated OD) stands for any clutter statistics. This approach has been investigated for uncorrelated noise and a further work would be to study the influence of the correlation on the detection performances for both known and unknown covariance matrix in exploiting the invariance property of the SIRV under linear transformation.
The approach that consists in evaluating the detectors performances with Padé approximation is being investigated and will be presented in a further work. The procedure is very easy to implement because of the use of the same expressions for P_{fa} and P_d. We have shown that the ouputs of the PEOD are positive random variate that is required for Padé method. More, the computational cost is insignificant if compared with numerical integration.

The Padé approximation method is so a very useful and efficient tool to deal with the problem and furthers works on the subject will complete the results.

References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, *National Bureau of Standard, AMS 55* (June 1964).

[2] H. Amindavar and J. A. Ritcey, *Padé Approximations for Detectability in K-Clutter and Noise*, *IEEE Trans.-AES* **30**–2 (April 1994) 424–434.

[3] H. Amindavar and J. A. Ritcey, *Padé Approximations of Probability Density Functions*, *IEEE Trans.-AES* **30**–2 (April 1994) 416–424.

[4] T. J. Barnard, and D. D. Weiner, *Non-gaussian clutter modeling with generalized spherically invariant random vectors*, *IEEE Trans.-SP* , **44**–10 (October 1996) 2384–2390.

[5] J.B. Billingsley, *Ground Clutter Measurements for Surface-Sited Radar*, *Technical Report 780, MIT*, February 1993

[6] A. Erdélyi, Tables of integral transforms, *Bateman manuscript project*, *California Institute of Technology McGraw-Hill* , **1-2**– (1954).

[7] F. Gini, M. V. Greco ,A. Farina and P. Lombardo, *Optimum and mismatched detection against K-distributed clutter plus gaussian clutter*, *IEEE Trans.-AES* , **34**–3 (July 1998) 860–876.

[8] E. Jay, J.P. Ovarlez and P. Duvaut, *New Methods of Radar Performances Analysis*, submitted to Signal Processing, July 1999

[9] J. J. K. Ó Ruanaidh, W. J. Fitzgerald, Numerical Bayesian Methods Applied to Signal Processing, *Statistics and Computing, Springer-Verlag, N.-Y.* (1996)

[10] J.P. Ovarlez and E. Jay, *New Methods of Radar Detection Performances Analysis*, *Proc.IEEE-ICASSP99* , Phoenix USA, March 1999

[11] M. Rangaswamy, D. Weiner and A. Öztürk, *Non-Gaussian Random Vector Identification Using Spherically Invariant Random Processes*, *IEEE Trans.-AES* **29**–1 (January 1993) 111–123

[12] M. A. Tanner, Tools for Statistical Inference, *Springer Series in Statistics, Springer-Verlag, N.-Y.* (1996)

[13] K. Yao, *A representation theorem and its application to spherically invariant random processes*, *IEEE Trans.-IT* **19**–2 (1973) 600–608.