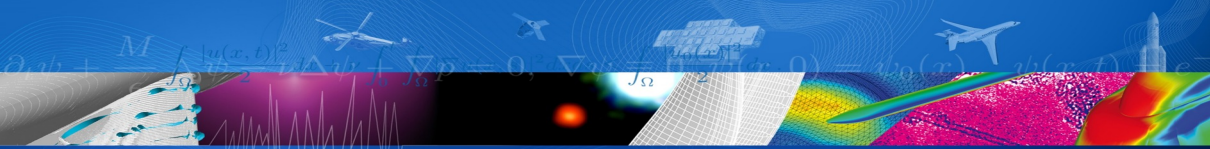

Time-Frequency Distributions for Non-Stationary Signal Analysis

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Plan

Outline

Fourier Analysis

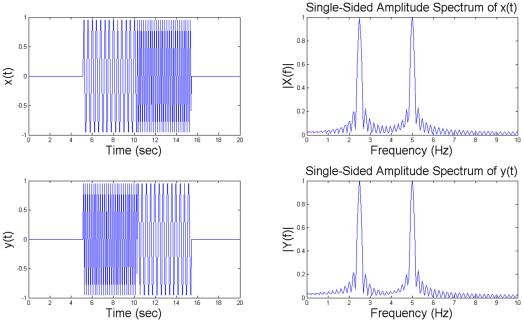
- **Fourier Analysis** is based on the universality of the concept of frequency:
  - physical waves: acoustic, vibrations, geophysic, optic, electromagnetism, etc.
  - periodicity of events: economy, biology, astronomy, etc.
- Its mathematical structure which naturally lends itself to common transformations such as linear filtering (complex exponential waves are eigenvectors of the convolution operator) by translating them in a very simple way.
- Fourier analysis has led to the development of numerous algorithms, processors and devices for frequency analysis.

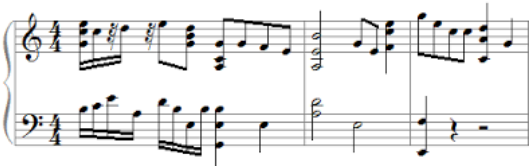
Fourier Transform: Signal decomposition onto eternal pure monochromatic waves basis

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, \exp\left(2i\,\pi\, f\, t\right) df \quad \text{where} \quad X(f) = \int_{-\infty}^{+\infty} x(t)\, \exp\left(-2i\,\pi\, f\, t\right) dt$$

Fourier analysis is well well adapted for **stationary signals**: signal whose spectral components do not vary with time (pure monochromatic waves, linear combination of pure monochromatic waves)

What Fourier Analysis misses





Non-Stationary Signals: signals whose spectral components vary with time

- Most of the information is carried by the non-stationarities: starting, end, transients
- Spectral analysis loses temporal information since it averages over time.
- Local analysis approach: adapt the stationary tools (TF, linear model) to the variations over time.

Some Solutions for Non-Stationary Signals Analysis

A solution: Parametric methods on time slices of the signal.

Examples of parametric spectral methods: AR, ARMA, MUSIC:

- Adaptive/evolving methods. Problem: need to estimate the order of the model on each local time slices
- A priori knowledge difficult to exploit on non-stationarities.

Alternative: Time-Frequency Analysis.

Global analysis of time-dependent spectral characteristics:

- The problem of estimating the model order is avoided. Problem: choosing the type of analysis. The analysis can play the role of a "model".
- Difficulty of interpretation of the analysis (uncertainty principle, interferences).

Outline

Instantaneous Frequency and Group Delay

First (simple) idea: define a local frequency (which depends on time)

- Write the signal in terms of modulated amplitude and phase (positive frequency analytical signal): $x(t) = a(t)\,\exp\left(i\phi(t)\right)$.
- Phase variations define the so-called **instantaneous frequency**: $f_x(t) = \dfrac{1}{2\,\pi}\dfrac{\partial\phi}{\partial t}$.

Examples

- **Monochromatic wave**: the instantaneous frequency of the signal $x(t) = \cos\left(2\pi\, f_0\, t\right)$ is $f_x(t) = f_0$ (constant).
- **Chirp** (linearly frequency modulated signal): the instantaneous frequency of the signal $x(t) = \cos\left(2\pi\left(2\,a\,t^2 + b\,t + c\right)\right)$ is $f_x(t) = a\,t + b$

Dual notion: Group Delay

the frequency variations of phase of $X(f) = A(f)\,\exp\left(i\,\psi(f)\right)$ define the **group delay** (time dependent on frequency):

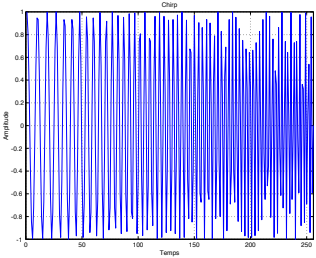
$$t_x(f) = -\frac{1}{2\,\pi}\frac{\partial\psi}{\partial f}.$$

Example: the signal $x(t) = \delta(t - t_0)$ as the group delay $t_x(f) = t_0$.

Presentation of Time Frequency Analysis

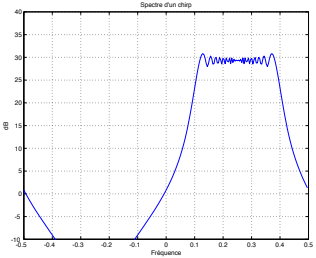
Linear Time-Frequency Analysis: Signal decomposition onto a dedicated basis

Time Analysis
(Shannon)



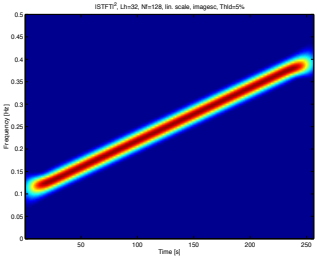
$$z(t) = \int_{-\infty}^{+\infty} z(u)\delta(u-t)\,du$$

Spectral Analysis
(Fourier)



$$z(t) = \int_{-\infty}^{+\infty} Z(v)\exp(2i\,\pi\,v\,t)\,dv$$

Linear Time-Frequency Analysis
(Gabor, Wavelets, Short Time)



$$z(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(u,v)\,g_{u,v}(t)\,du\,dv$$

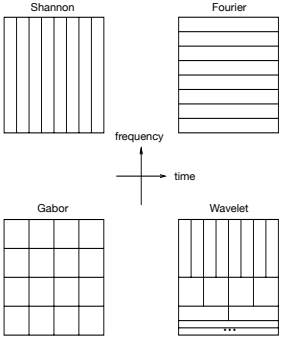
Linear Time-Frequency Analysis and Heisenberg Inequality

Heisenberg Inequality: For all signal window $h(.)$ and defining the spectral bandwidth σ_f^h and the time duration σ_t^h, we have:

$$\sigma_f^h \, \sigma_t^h \geq 1$$

The Gaussian shape signal is the only signal verifying the above equality ($\sigma_f^h \, \sigma_t^h = 1$).

Good time resolution: $\sigma_t = 0$
Bad frequency resolution:
$\sigma_f = \infty$



Bad time resolution: $\sigma_t = \infty$
Good frequency resolution:
$\sigma_f = 0$

joint time-frequency resolutions:
$\sigma_t = \sigma_t^h$ and $\sigma_f = 1/\sigma_t^h$

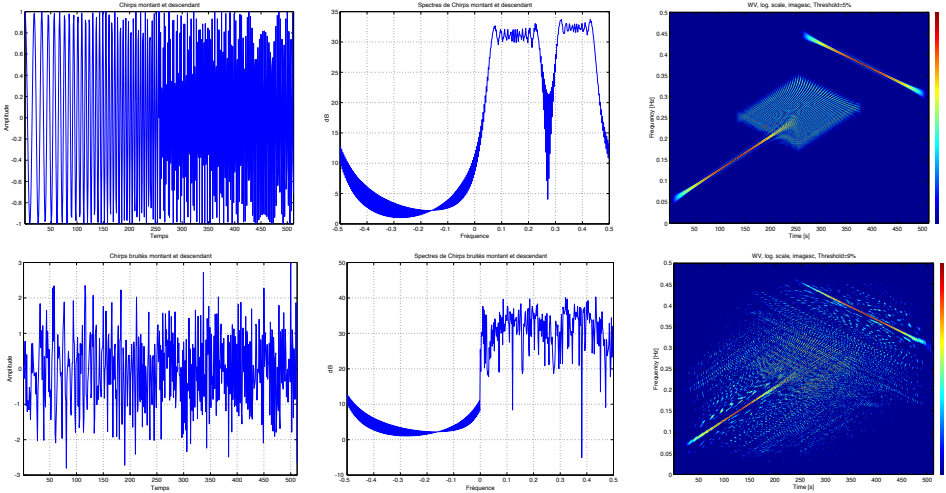
$\sigma_f^h \, \sigma_t^h = 1$

joint time-frequency resolutions:
$\sigma_t = \sigma_t^h / f$ and $\sigma_f = f / \sigma_t^h$

$\sigma_f^h \, \sigma_t^h = 1$

Advantages of the Time-Frequency Analysis

Readability, access to instantaneous frequency or group delay of signals, robust to noise



Building Quadratic Time-Frequency Distributions

- Theory of operators: mean value of operator (observable) like in Quantum Mechanics Theory.

☞ **Theory of multivariate probability density functions, Theory of copulas**.

Goal: Designing a joint time-frequency distribution function $P(t, f)$ which

- has to distribute the total energy E_z of the signal $z(.)$ in the whole Time-Frequency plane:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(t, f)\, dt\, df = E_z = \int_{-\infty}^{+\infty} |z(t)|^2\, dt = \int_{-\infty}^{+\infty} |Z(f)|^2\, df$$

- has to retrieve the following marginal distribution functions:

$$\int_{-\infty}^{+\infty} P(t, f)\, dt = |Z(f)|^2 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} P(t, f)\, df = |z(t)|^2$$

Required Properties 1/2

- **Covariance Principle by action of a group**: Consistency of the distribution P_z after a physical transformation group $\mathcal{T}$ applied on the signal $z(.)$ and on its dual representation $\mathcal{T}' P_z$

$$\begin{array}{ccc} z & \longrightarrow & P_z \\ \mathcal{T}\downarrow & & \mathcal{T}'\downarrow \\ \mathcal{T}z & \longrightarrow & \mathcal{T}' P_z = P_{\mathcal{T}z} \end{array}$$

  Examples: time translation, frequency translation (approximate Doppler effect), scaling (true Doppler effect), rotation, Lorentz, etc.

- **Unitarity** (Conservation of the inner product, Moyal Formula, Extended Parseval Theorem):

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_{z_1}(t,f)\, P_{z_2}(t,f)\, dt\, df = \left| \int_{-\infty}^{+\infty} z_1(t)\, z_2^*(t)\, dt \right|^2$$

Required Properties 2/2

- Access to the instantaneous frequency $f_z(.)$ and group delay $t_z(.)$ through marginalization:

$$f_z(t) = \frac{1}{2\,\pi}\frac{\partial \phi}{\partial t} = \frac{\displaystyle\int_{-\infty}^{+\infty} f\,P(t,f)\,df}{\displaystyle\int_{-\infty}^{+\infty} P(t,f)\,df} \qquad \text{and} \qquad t_z(f) = -\frac{1}{2\,\pi}\frac{\partial \psi}{\partial f} = \frac{\displaystyle\int_{-\infty}^{+\infty} t\,P(t,f)\,dt}{\displaystyle\int_{-\infty}^{+\infty} P(t,f)\,dt},$$

- Positivity,
- Conservation of spectral and time signal supports,
- Perfect time-frequency localization on some particular signals:
  - Monochromatic signals: $z(t) = \exp(2i\,\pi f_0\,t) \longrightarrow P_z(t,f) = \delta(f - f_0)$,
  - Shock type signals: $z(t) = \delta(t - t_0) \longrightarrow P_z(t,f) = \delta(t - t_0)$,
  - Monocomponent frequency modulated signals:

$$z(t) = \exp(i\phi(t)) \longrightarrow P_z(t,f) = \delta\left(f - f_z(t)\right) = \delta\left(t - \frac{1}{2\,\pi}\frac{\partial \phi}{\partial t}\right).$$

How to Build the Quadratic Time-Frequency Distributions

Building through kernel type model respecting as many required properties as possible

In order to preserve the primary notion of energy distribution in the time-frequency plane, these distributions are constructed on bilinear forms $z(u)\,z^*(v)$ of the time signal (or, equivalently, frequency $Z(u)\,Z^*(v)$):

$$\begin{aligned} P(t,f) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(u,v;t,f)\,z(u)\,z^*(v)\,du\,dv \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{K}(u,v;t,f)\,Z(u)\,Z^*(v)\,du\,dv \end{aligned}$$

The kernel $K(u,v;t,f)$ (or $\hat{K}(u,v;t,f)$) has to be determined according to the required properties.

Outline

Weyl-Heisenberg group and the Cohen's class

Determination of the class of covariant solutions through the time and frequency translation transformation group (Weyl-Heisenberg):

$$\begin{array}{ccc} z(t) & \longrightarrow & P_z(t,f) \\ \mathcal{T}\big\downarrow & & \mathcal{T}'\big\downarrow \\ z'(t) = z(t-t_0)\exp(2i\,\pi\,f_0\,t) & \longrightarrow & P_{z'}(t,f) = P_z(t-t_0, f-f_0) \end{array}$$

Direct solutions: Cohen's class parametrized by a 2D kernel $\psi(.,.)$ (or equivalently $\Pi(.,.)$):

$$\begin{aligned} P_z(t,f) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2i\pi\xi(s-t)}\,\psi(\xi,\tau)\,z\left(s+\frac{\tau}{2}\right)\,z^*\left(s-\frac{\tau}{2}\right)\,e^{-2i\pi f\tau}\,d\xi\,ds\,d\tau \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Pi(s-t,\xi-f)\,W_z(s,\xi)\,ds\,d\xi \end{aligned}$$

where the central distribution W_z is the well known **Wigner-Ville distribution**:

$$W_z(t,f) = \int_{-\infty}^{+\infty} z\left(t+\frac{\tau}{2}\right)\,z^*\left(t-\frac{\tau}{2}\right)\,e^{-2i\pi f\tau}\,d\tau$$

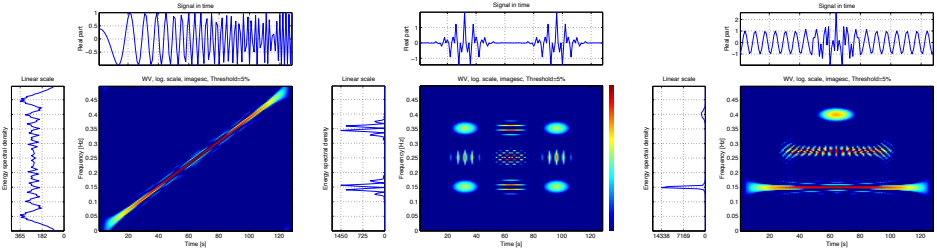
Some distributions of the Cohen's class

Nom	$\psi(\xi,\tau)$	$P_z(t,f)$
Wigner-Ville	**1**	$\int_{-\infty}^{+\infty} z\left(t+\frac{\tau}{2}\right) z^*\left(t-\frac{\tau}{2}\right) e^{-2i\pi f\tau}\, d\tau$
s-Wigner	$e^{2i\pi s\xi\tau}$	$\int_{-\infty}^{+\infty} z\left(t-\left(s-\frac{1}{2}\right)\tau\right) z^*\left(t-\left(s+\frac{1}{2}\right)\tau\right) e^{-2i\pi f\tau}\, d\tau$
Rihaczek	$e^{i\pi\xi\tau}$	$z(t)\, Z^*(f)\, e^{-2i\pi f\tau}$
Born-Jordan	$\frac{\sin \pi\xi\tau}{\pi\xi\tau}$	$\int_{-\infty}^{+\infty} \left[\frac{1}{\lvert\tau\rvert}\int_{t-\lvert\tau\rvert/2}^{t+\lvert\tau\rvert/2} z\left(s+\frac{\tau}{2}\right) z^*\left(s-\frac{\tau}{2}\right) ds\right] e^{-2i\pi f\tau}\, d\tau$
Choï-Williams	$e^{-(\pi\xi\tau/\sigma)^2/2}$	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma}{\lvert\tau\rvert} e^{-2\sigma^2(s-t)^2/\tau^2} z\left(s+\frac{\tau}{2}\right) z^*\left(s-\frac{\tau}{2}\right) e^{-2i\pi f\tau}\, ds\, d\tau$
Spectrogramme	$A_h(\xi,\tau)$	$\left\lvert\int_{-\infty}^{+\infty} z(s)\, h^*(s-t)\, e^{-2i\pi fs}\, ds\right\rvert^2$
Séparable	$G(\xi)\, h(\tau)$	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\tau)\, g(s-t)\, z\left(s+\frac{\tau}{2}\right) z^*\left(s-\frac{\tau}{2}\right) e^{-2i\pi f\tau} ds\, d\tau$

Main Quadratic Time-Frequency Distribution: Wigner-Ville

- Wigner-Ville [E. P. Wigner, 1932]: main and most popular distribution:

$$W_z(t,f) = \int_{-\infty}^{+\infty} z\left(t+\frac{\tau}{2}\right) z^*\left(t-\frac{\tau}{2}\right) e^{-2i\pi f\tau}\, d\tau = \int_{-\infty}^{+\infty} Z\left(f+\frac{\nu}{2}\right) Z^*\left(f-\frac{\nu}{2}\right) e^{-2i\pi t\nu}\, d\nu$$

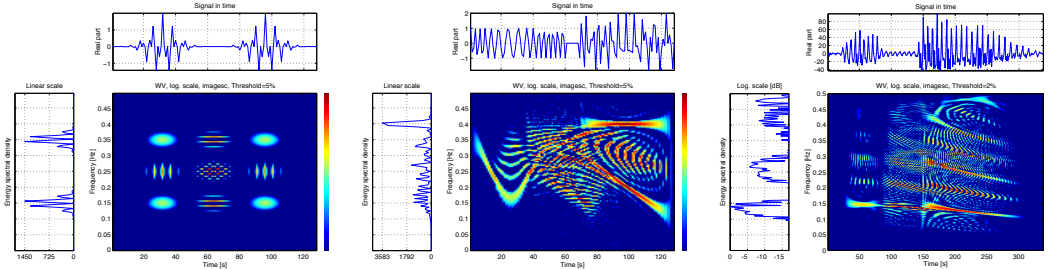


- Link to Woodward's Radar Ambiguity Function $A_z(\tau,\nu)$ through Fourier Transform

$$A_z(\tau,\nu) = \int_{-\infty}^{+\infty} z\left(u+\frac{\tau}{2}\right) z^*\left(u-\frac{\tau}{2}\right) e^{-2i\pi\nu u}\, du = \int_{-\infty}^{+\infty} W_z(t,f)\, e^{-2i\pi(t\nu+f\tau)}\, dt\, df$$

Some Drawbacks

- No distribution satisfies all the constraints (e.g. positivity and unity).
- Illegibility due to interference between elementary signal components
- Need to *erase* or reduce interferences between each time and spectral components

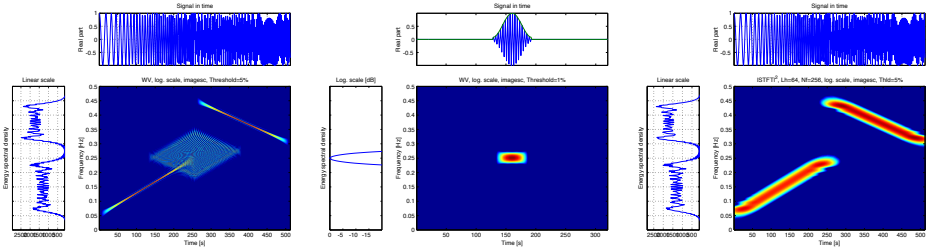


Another Well-Known Time-Frequency Distribution: Spectrogram

- Regularized version of the Wigner-Ville distribution: smoothing operation to reduce interferences between components (ex: Short Time Fourier Transform, Gabor)

$$P_l(t_0, f_0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_z(t, f)\, P_h(t - t_0, f - f_0)\, dt\, df = \left| \int_{-\infty}^{+\infty} z(u)\, h^*(u - t_0)\, e^{-2i\pi\, f_0\, u}\, du \right|^2$$

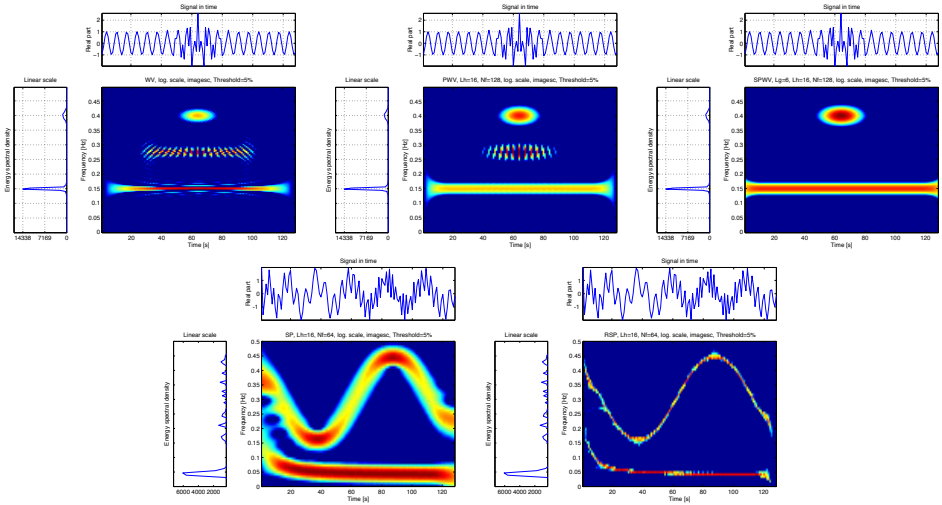
where $h(t)\, \exp(2i\pi\, f_0\, t)$ is an analyzing window localized around $t = 0$ and $f = f_0$.



$P_z(t, f)$ $\qquad$ $P_h(t - t_0, f - f_0)$ $\qquad$ $P_l(t_0, f_0)$

- Drawback: Heisenberg inequality, no control on joint time and spectral resolutions

Another Quadratic Time-Frequency Distributions



Outline

Going further with Affine Group

Determination of the class of covariant solutions through the time-scale and time-shift transformation group (Affine group):

$$\begin{array}{ccc} Z(f) & \longrightarrow & P_Z(t,f) \\ \mathcal{T}\big\downarrow & & \mathcal{T}'\big\downarrow \\ Z'(f) = a^{r+1}\, Z(a\, f)\, \exp\left(2i\,\pi\, t_0\, f\right) & \longrightarrow & P_{Z'}(t,f) = a^q\, P_Z\left(a^{-1}\,(t-t_0)\,,\, a\, f\right) \end{array}$$

Direct solutions: Bertrand's class parametrized by a 2D kernel $K(u,v)$:

$$P_Z(t,f) = \int_0^{+\infty}\int_0^{+\infty} K(u,v)\, Z(f\,u)\, Z^*(f\,v)\, e^{2i\pi\, f\, t\,(u-v)}\, du\, dv$$

Example of the central distribution: **Bertrand's Unitary Affine Distribution**:

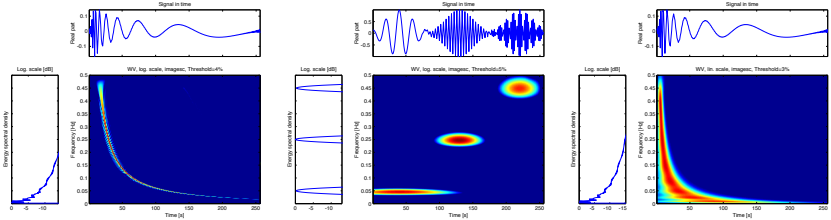
$$P_Z^0(t,f) = f^{2r+2-q} \int_{-\infty}^{+\infty} \left[\frac{u}{2\,\sinh(u/2)}\right]^{2r+2} Z\left(f\,\frac{u\,e^{u/2}}{2\,\sinh(u/2)}\right) Z^*\left(f\,\frac{u\,e^{-u/2}}{2\,\sinh(u/2)}\right) e^{2i\pi\, f\, t\, u}\, du$$

Going further with Affine Group

- Regularized version of the Unitary Affine distribution: smoothing operation to reduce interferences between components (**Wavelet Transform**)

$$\tilde{P}(t_0, f_0) = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} P_Z^0(t,f)\, P_\Psi^0\left(\frac{f}{f_0}(t-t_0), \frac{f_0}{f}\right) dt\, df = \left| \int_{-\infty}^{+\infty} Z(f)\, \Psi^*\left(\frac{f}{f_0}\right) e^{-2i\pi f t_0}\, df \right|^2$$

where $\Psi(f)\, \exp(2i\pi f t)$ is an analyzing mother window localized around $t = 0$ and $f = 1$.

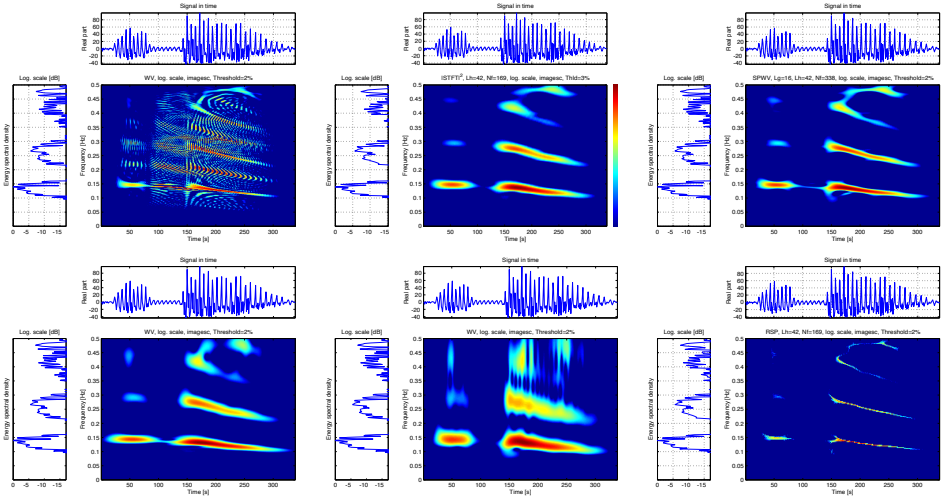


$P_Z^0(t,f)$ $\quad$ $P_\Psi^0\left(\frac{f}{f_0}(t-t_0), \frac{f_0}{f}\right)$ $\quad$ $\tilde{P}(t_0, f_0)$

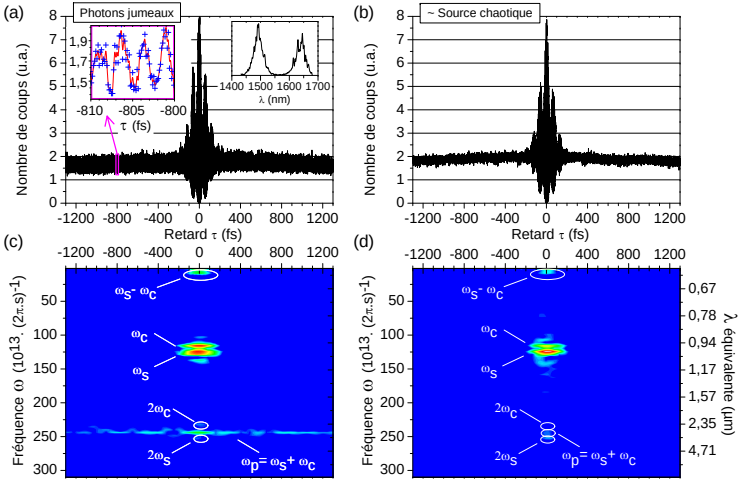
- Drawback: Heisenberg inequality, no control on joint time and spectral resolutions

Outline

Applications: Speech Analysis

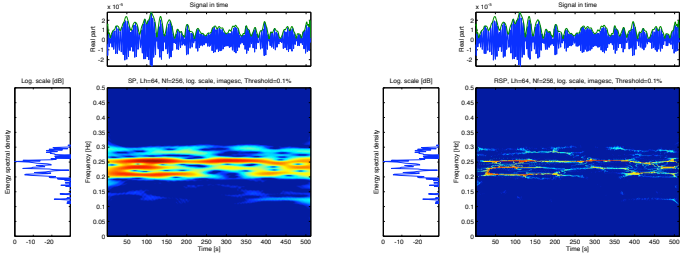


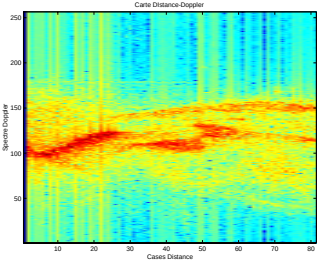
Applications: Two-Photon-Counting Interferometry

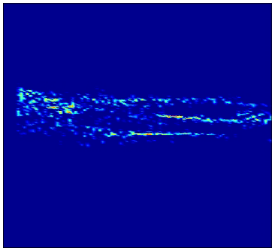


F. Boitier, A. Godard, N. Dubreuil, P. Delaye, C. Fabre, and E. Rosencher, *Nature Communications* **2**, 425 (2011).

Applications: Analysis of Range-Doppler Map of Ionosperic Clutter

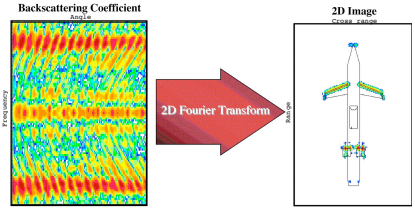






Applications: Radar Imaging

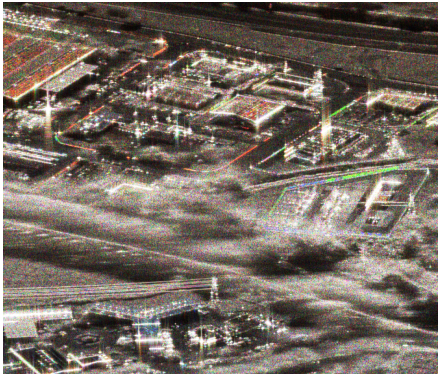
- Conventional Fourier Imaging (laboratory, SAR, ISAR) does not exploit the potential non-stationarities of the scatterers
- Hypothesis of bright points modeling: all the scatterers localized in $\mathbf{x}$ and characterized by the complex spatial amplitude distribution $I(\mathbf{x})$ have **the same behavior** for any wave vector $\mathbf{k} = \frac{2f}{c}(\cos\theta, \sin\theta)^T$.
  After some processing, the backscattering coefficient $H(\mathbf{k})$ acquired by the radar is simply related to the SAR image $I(\mathbf{x})$ through:
  $$H(\mathbf{k}) = \int_{\mathcal{D}_{\mathbf{x}}} I(\mathbf{x}) \exp\left(-2\,i\,\pi\,\mathbf{k}^T\,\mathbf{x}\right) d\mathbf{x}$$
- The SAR image $I(\mathbf{x})$ is then obtained through the Inverse Fourier Transform:
  $$I(\mathbf{x}) = \int_{\mathcal{D}_{\mathbf{k}}} H(\mathbf{k}) \exp\left(2\,i\,\pi\,\mathbf{k}^T\,\mathbf{x}\right) d\mathbf{k}$$



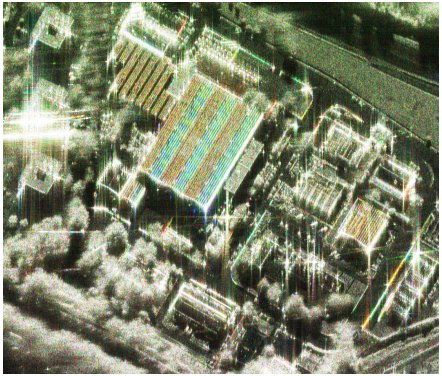
With this model, all information relative to frequency f and angle θ are lost. Hence, spectral and angular diversities are lost.

Applications: Radar Imaging

elevation 30°



elevation 50°



Sub-band 1 Sub-band 2 Sub-band 3

Scatterers have different behaviors with respect to the frequency and illumination direction.

Applications: Radar Imaging

Time-Frequency Distributions are generally devoted to non-stationary time signals analysis. They can be easily extended in 2D.

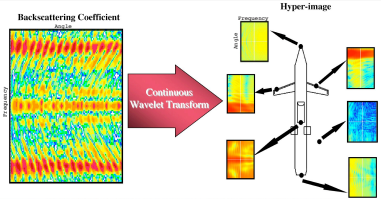
Key idea: In the context of SAR Imaging, Time-Frequency Analysis allows:

- to highlight the coloration and anisotropy properties of monodimensional SAR scatterers,
- to characterize each pixel of the complex SAR image with a vector of information related to angular or/and frequency behaviors.

Time Frequeny Analysis and the physical group theory (Heisenberg or affine group) allow to construct **hyperimages** [Bertrand 91, Ovarlez 92] through:

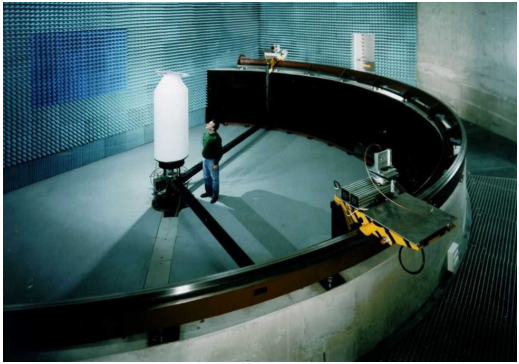
$$\tilde{I}(\mathbf{r}_0, \mathbf{k}_0) = < H(.), \Psi_{\mathbf{r}_0,\mathbf{k}_0}(.) > = \int_{\mathcal{D}_\mathbf{k}} H(\mathbf{k})\, \Psi^*_{\mathbf{r}_0,\mathbf{k}_0}(\mathbf{k})\, d\mathbf{k}\,,$$

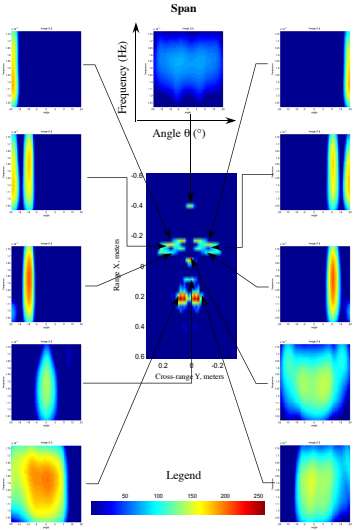
where $\Psi_{\mathbf{r}_0,\mathbf{k}_0}(\mathbf{k})$ is a family of wavelet bases (Gabor, wavelet) generated from a mother wavelet $\phi(f, \theta)$ through the chosen physical group of transformation (translations, scale in frequency, etc.) and where $\mathcal{D}_\mathbf{k}$ is the spectral/angular support of Ψ.



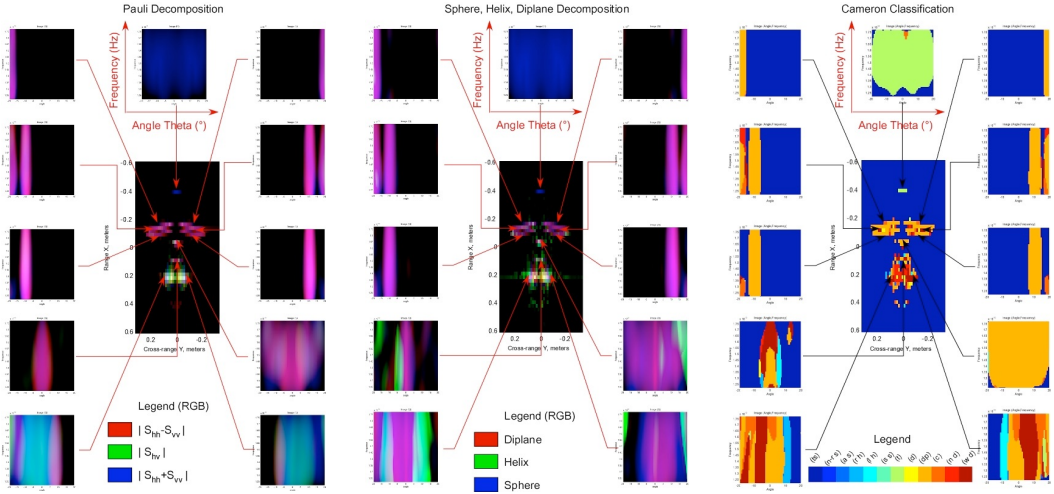
Applications: Radar Imaging [M. Tria 05]

Anechoic room geometry



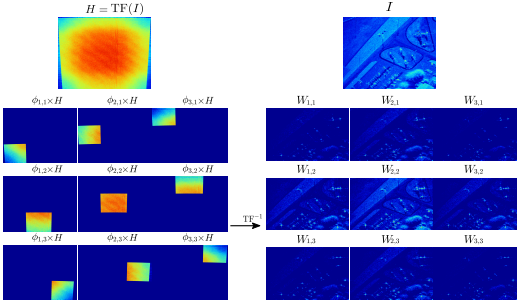


Applications: Polarimetric Radar Imaging [M. Duquenoy 09]



Applications: Detection in SAR image, SAR Change Detection

From Mono-Channel to Multi-Channel SAR Image: Example of $N_f = 3$ sub-bands and $N_\theta = 3$ sub-looks image decomposition:



Exploitation of the diversity for target detection and change detection

Each pixel i of the mono-channel SAR image can now be characterized by a N-vector $\mathbf{x}_i = \left[W^i_{1,1}, \ldots, W^i_{N_f,N_\theta}\right]^T$ of information ($N = N_f\, N_\theta$) related to **dispersion** in frequency domain and **anisotropy** in angular domain [P. Formont 13, A. Mian 19].