On the false alarm probability of the Normalized Matched Filter for off-grid target detection

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Radar Detection Context

The classical Radar detection problem is the following binary Hypothesis Test:

$$\begin{cases} H_0 : \mathbf{r} = \mathbf{n} \\ H_1 : \mathbf{r} = \alpha\, \mathbf{d}(\theta) + \mathbf{n} \end{cases}, \text{ where}$$

- $\mathbf{r} \in \mathbb{C}^N$ is the observation,
- $\mathbf{d}(\theta) \in \mathbb{C}^N$ is the signal echo reflected by a target with parameters θ (range, angle, Doppler...),
- $\alpha \in \mathbb{C}$ is the complex amplitude of the received signal,
- $\mathbf{n} \in \mathbb{C}^N$ is the additive noise vector, independent of the source signal. Our results hold for any spherically invariant distribution such as $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma^2\, \mathbf{\Gamma})$.

$\mathbf{d}(\theta)$: General spectral analysis model (angle or Doppler with Radar) :

$$\mathbf{d}(\theta) = \frac{1}{\sqrt{N}} \left[1, e^{2i\pi\theta}, \ldots, e^{2i\pi(N-1)\theta}\right]^T .$$

- When the unknown parameters are σ, α under H_1 and σ under H_0, with θ known, we use the following Normalized Matched Filter (NMF) to decide which hypothesis we are under[1]:

$$t_{\mathbf{\Gamma}}(\mathbf{r}, \theta) = \left|\mathbf{s}(\theta)^H\, \mathbf{u}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta.$$

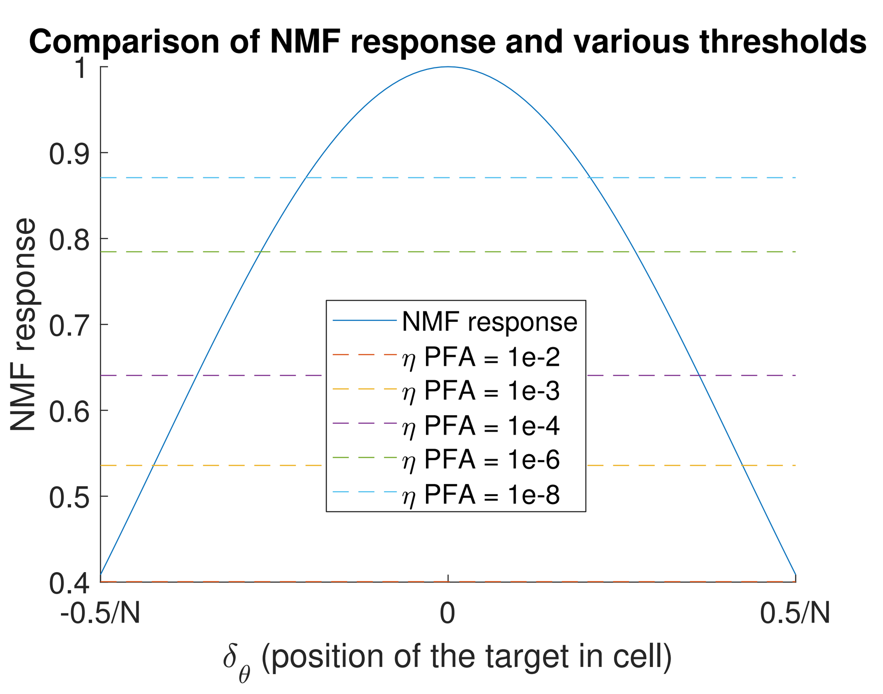
where $\mathbf{s}$ and $\mathbf{u}$ are the whitened normalized versions of $\mathbf{d}$ and $\mathbf{r}$, on $\mathbb{S}^{N-1}$.

Mismatch And Off-Grid Issue

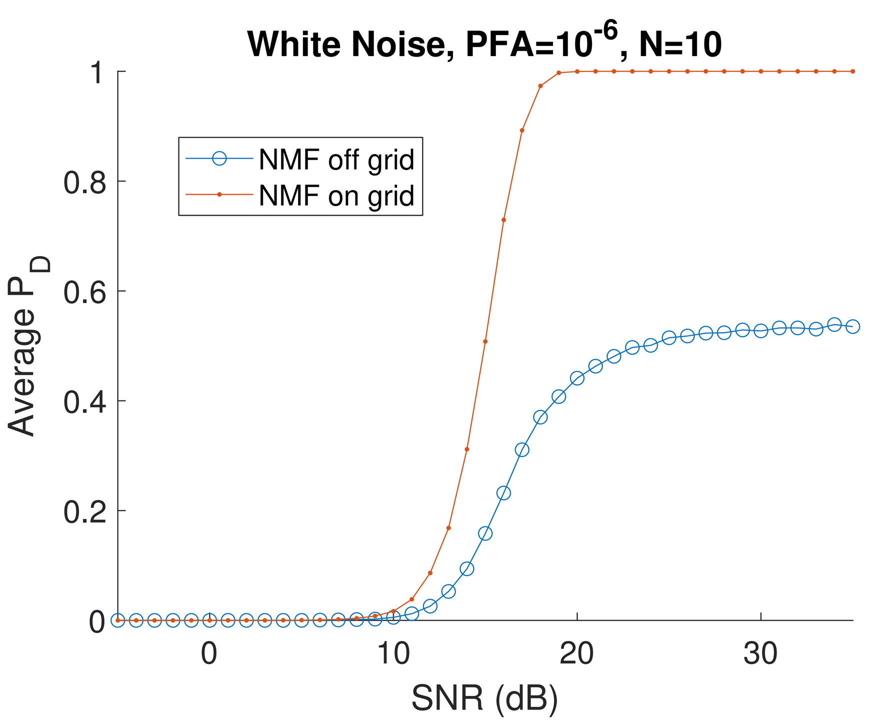
- The NMF is an angle detector; it is equal to the square of the cosinus of the angle between $\mathbf{s}$ and $\mathbf{u}$.
- If there is mismatch between the reference signal $\mathbf{s}(\theta_0)$ and target signal $\mathbf{s}(\theta)$, the NMF response is degraded even if the power level of the signal is very high [4][5].
- Since the parameter of the target θ is unknown in reality, we usually test for fixed values of θ_0 in a grid $\mathbf{G}$.
- This induces a mismatch that degrades detection performances.
- In order to fight off the mismatch, the optimal known solution is to use the following GLRT detector:

$$GLRT(\mathbf{r}, \mathcal{D}) = \max_{\theta_c \in \mathcal{D}} \left|\mathbf{s}(\theta_c)^H \mathbf{u}\right| \underset{H_0}{\overset{H_1}{\gtrless}} w'$$

- We derive its P_{FA}-threshold relationship, unknown in the litterature.



NMF response as a function of the mismatch with various thresholds



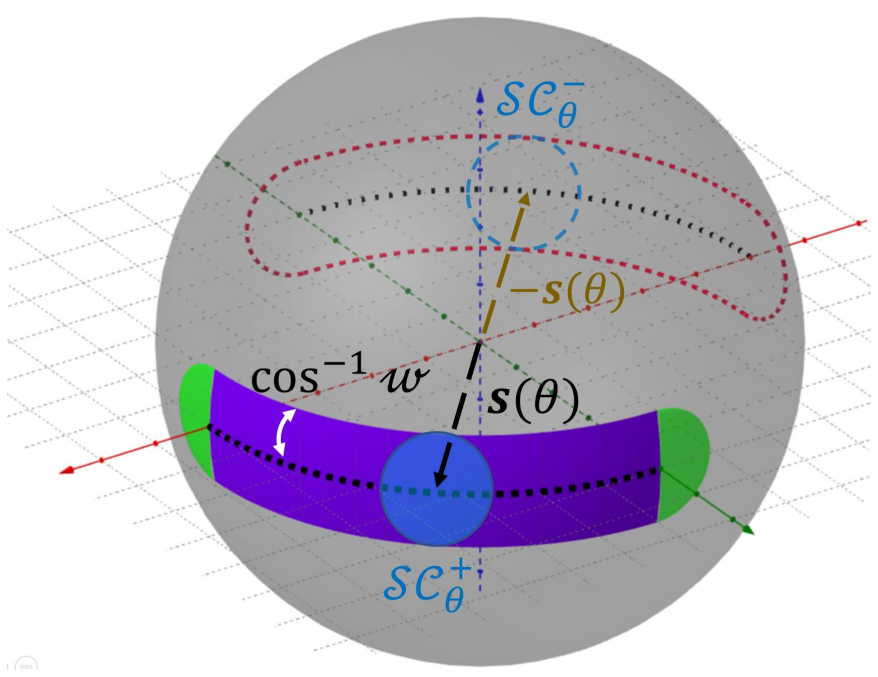
Probability of Detection of the NMF as a function of SNR in on-grid and off-grid case

Methodology and Result

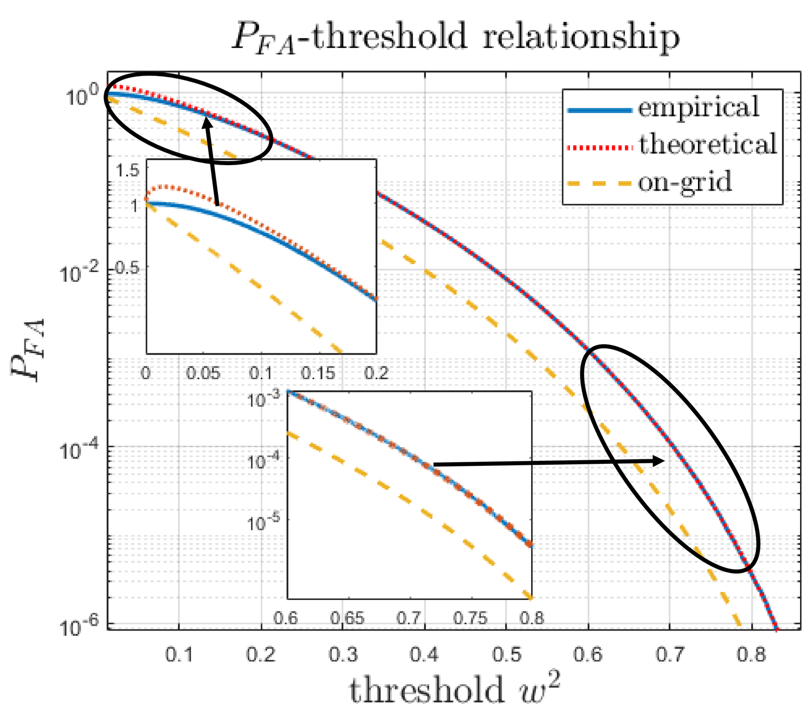
- To find the P_{FA}, we use a geometrical approach.
- The P_{FA} of the detector is equal to the ratio between the acceptance region and the surface of the unit sphere.
- For real signals, the acceptance region is a tube composed of the spherical caps around $\mathbf{s}(\theta_c)$ for each θ_c in the search domain $\mathcal{D}$. In [2], Hotteling gives a formula for computing the surface of such tubes.
- Our signals are complex and so this result is not applicable. We show our problem reduces to finding the surface of a tube around a 2D manifold. [3] enables us to compute the surface of the tube in our problem.
- We get, under white noise ($\mathbf{\Gamma} = \mathbf{I}$):

$$P_{FA} = \overbrace{(1 - w^2)^{N-1}}^{\text{on-grid relation}} + \sqrt{\frac{\pi}{3}}\, \frac{\Gamma(N)\, w\, (1 - w^2)^{N - \frac{3}{2}}}{\Gamma\left(N - \frac{1}{2}\right)}\, (N^2 - 1)^{\frac{1}{2}} \underbrace{(\theta_2 - \theta_1)}_{\text{length of search domain}} .$$

- Under colored noise, we propose an integral that can be computed numerically.



Acceptance zone for real signals



Numerical validation of our result

Numerical Results

- We tested our results under white noise against numerical threshold obtained through Monte Carlo simulations approximating the GLRT with 30 tests per resolution cell.
- It fits perfectly for all P_{FA} of practical interest.
- For really high P_{FA}, the tube overlaps itself and the formula is an upper bound (subject of a future journal article).

Références

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